Rational Transfer functions and the Bode Plot
The transfer function

- The transfer function is the Fourier transform of the impulse response
- Filters we can make have a rational transfer function: the transfer function is a ratio of two polynomials with real coefficients.

(strictly speaking this is called the “Padé approximation”: it states that any real function can be approximated by a rational function. The higher the degree of the polynomials the closer the approximation can be made)

The notation is \( s = j\omega \). The signals assumed to be sinusoid:

\[
V = V_0 e^{j\omega t + \phi}
\]

\[
H(s) = \frac{P_n(s)}{Q_n(s)} = \frac{\sum_{k=0}^{n} a_k s^k}{\sum_{k=0}^{m} b_k s^k} = \frac{a_n(s-z_1)(s-z_2)\cdots(s-z_n)}{b_m(s-p_1)(s-p_2)\cdots(s-p_m)}
\]

- The roots \( z_k \) of the numerator polynomial are called the “zeroes” of \( H \)
- The roots \( p_k \) of the denominator polynomial are called the “poles” of \( H \)
- The pole positions on the complex frequency plane entirely determine the filter properties.
- Note that since \( s = j\omega \) the denominator is seldom zero unless it has pure imaginary roots
The Bode plot – some maths

Complex numbers can be written in rectangular or polar representation:

\[ z = x + iy = \text{Re}^{j\theta} \]

The natural logarithm function works correctly with complex argument:

\[ \ln z = \ln \left( \text{Re}^{j(\theta + 2n\pi)} \right) = \ln R + j\theta = \ln |z| + j\arg z + j2\pi n \]

It follows that, if we restrict the phase to a circle,

\[ \text{Im} \ln z = \arg z \]

Since \( H(s) \) is rational, we can write:

\[
\ln H(s) = \ln \left( \frac{a_n \prod_{k=1}^{n} (s - z_k)}{b_m \prod_{k=1}^{m} (s - p_k)} \right) = \ln \frac{a_n}{b_m} + \sum_{k=1}^{n} \ln (s - z_k) - \sum_{k=1}^{m} \ln (s - p_k)
\]
The Bode plot – Magnitude plot

Since \( H(s) \) is rational, we can write:

\[
\ln |H(s)| = \ln \left| \frac{a_n}{b_m} \right| + \left( \sum_{k=1}^{n} \ln |s - z_k| \right) - \left( \sum_{k=1}^{m} \ln |s - p_k| \right)
\]

The log of the transfer function is the sum of terms of the form \( \ln |s - x| \)
Each term takes two simple limits for high and low frequencies respectively:

\[
\lim_{|s|>>x} \ln |s - x| = \ln |s| = \ln \omega, \quad \lim_{|s|<<x} \ln |s - x| = \ln |x|
\]

Each of the terms in the numerator contributes a constant at low frequencies and a line of slope 1 at high frequencies. The corner is at \( \omega = z_k \)

Each of the terms in the denominator contributes a constant at low frequencies and a line of slope -1 at high frequencies. The corner is at \( \omega = p_k \)
The Bode plot – Phase plot

Phase plot: Make a linear – log plot of the phase (i.e. the imaginary part of the logarithm of the transfer function) versus the log of frequency.

\[ \arg H(s) = \Im \ln H(s) = \arg \frac{a_n}{b_m} + \sum_{k=1}^{n} \arg(s - z_k) - \sum_{k=1}^{m} \arg(s - p_k) \]

Each term in the numerator contributes (remember \( s \) is imaginary!)

\[ \lim_{\omega \to 0} \arg(s - z_n) = 0 \]
\[ \lim_{\omega \to \infty} \arg(s - z_n) = \pi / 2 \]
\[ \lim_{\omega \to z_n} \arg(s - z_n) = \tan^{-1} \left( \frac{\omega}{z_n} \right) \approx \frac{\pi}{4} + \frac{1}{2} \left( \frac{\omega}{z_n} - 1 \right) \approx \frac{\pi}{4} + \frac{1}{2} \ln \left( \frac{\omega}{z_n} \right) \]

since \( \ln(1 + \epsilon) \approx \epsilon \). in this case, \( \epsilon = \frac{\omega}{z_n} - 1 \). also note that \( \log x = \ln x / \ln 10 \)

- each term has linear asymptote at 0 radians for small frequencies
- each term has linear asymptote at \( \pi/2 \) radians at high frequencies
- Each term contributes a linear (-/+) slope near the pole or zero
- a factor of 10 in frequency adds/subtracts 45 degrees to phase
Bode plot example - poles

Magnitude plot:
Break point at pole
Slope: -20dB per decade increase in frequency

Phase plot:
-45 degrees at pole
Slope:
-45 degrees per factor of 5 increase in frequency

Often approximated to
-90 degrees/ 2 decades
Bode plots - Zeros

Magnitude plot:
- Break point at pole
- Slope: 20dB per decade increase in frequency

Phase plot:
- 45 degrees at pole
- Slope:
  - 45 degrees per factor of 5 increase in frequency

Often approximated to
- 90 degrees/2 decades
Bode plot example: 2 poles 1 zero

\[ H(f) = \frac{(s + 1)}{s(s + 10)} \]