6 Frequency Measurement and manipulation

By the end of this section you will be able to:

- Describe and quantify the operation of frequency measuring instruments.
- Describe a phase locked loop, and its principles of operation.
- Compute the centre frequency, capture and tracking range of a phase locked loop
- Describe several methods for frequency multiplication.

6.1 Frequency as a state variable

The generation and measurement of frequency is related to the ancient problem of determining time by building clocks. The importance of clocks to navigation on earth, on the sea and recently in air and in space means that enormous resources have been allocated over the centuries in resolving it. In our days time measurement is by far the most exact branch of metrology, and time is known (at least to the Standards organisations) with an estimated error of 1 part in $10^{14}$. Techniques to further improve time determination by another 2 orders of magnitude are known and are being implemented. Suffice it to say that the theoretical accuracy of time determined by GPS is better than 1 part in $10^8$ which corresponds to an uncertainty of a fraction of a second/year.

It is counter intuitive that the frequency of a signal can be considered as a state variable, a quantity which not only can be measured, but also can be manipulated.

![Figure 6-1: Historical development of clock accuracy (after HP application note 1289: The science of timekeeping)](image-url)
Figure 6-2: Relative accuracy and stability of different clocks (after HP application note 1289: The science of timekeeping)

6.2 Definitions

A periodic signal $S$ with period $T$ is defined by:

$$S(t + nT) = S(t), \forall n \in \mathbb{Z}$$  \hspace{1cm} (1)

Such a signal possesses a discrete Fourier transform, or a spectrum consisting of a (perhaps infinite) number of delta functions. The fundamental frequency or, simply, the frequency of the signal is:

$$f_0 = \frac{1}{T}$$  \hspace{1cm} (2)

The other frequency components of the signal are at integral multiples, or harmonics, of this fundamental frequency:

$$f_n = n f_0$$  \hspace{1cm} (3)

No real signal has an infinite duration, so that no real signal is strictly speaking periodic. This implies (by Fourier analysis) that a signal of duration $T_0$ has a spectrum consisting of finite width impulses around the positions of the spectral lines of a similar signal of infinite extent. The width of each spectral impulse is roughly:

$$\Delta f = \frac{1}{T_0}$$  \hspace{1cm} (4)

We have established a very fundamental theorem, one related to the famous uncertainty principle in physics: A perfect frequency measurement requires an infinite time of observation; as such it is impossible, even in complete absence of noise.
Nonetheless, we will assume that a perfect frequency measurement is possible and apply any corrections as required.

The phase of a signal is an index to the signal's value over one period, or for times between:

\[ nT \leq t \leq (n+1)T \Rightarrow t = \left( \frac{n + \phi}{2\pi} \right)T \]  

Conventionally we measure phase in angle units (radians in 4.5)

Like all measurements, measurement of the frequency of a signal is a judicious application of the definition of the concept of frequency. The question asked when performing a frequency measurement is how often a particular feature of the signal repeats itself. The zero crossing i.e. the time the signal crosses its average value, is the feature most commonly used for frequency measurement. For this reason a limiting amplifier, converting the signal to a square wave, usually precedes frequency measuring instrument.

The measurement of frequency is a non-linear mathematical operation. If we assume the signal is the pure “sinusoid”, the favourite signal in introductory courses, then we can see that:

\[ V(t) = V_o e^{|\omega t + \phi|} = V_o e^{j\omega t + \phi} \Rightarrow \omega t + \phi = \text{Im} \left( \ln V(t) \right) \Rightarrow \]

\[ \omega = \frac{d}{dt} \text{Im} \left( \ln V(t) \right) \]

\[ \phi = \Phi \mod 2\pi \]

This reasoning clarifies why, for example, the general analysis of the Phase locked loop is an almost intractable nonlinear problem, and why there are ambiguities in the generation and measurement of signals and their frequencies.

### 6.3 Frequency generation - Oscillators

Periodic signals are generated by oscillators, so we include a brief review of the theory and practice of oscillators here. Later in this chapter we examine how we can manipulate the output of an oscillator to obtain any desired frequency.

There is an old motto between radio engineers that “every amplifier oscillates and every oscillator amplifies”. This is presumably Murphy’s law applied to electronics, but it also indicates a very fundamental truth: Oscillators are amplifiers, and amplifiers are oscillators. The difference between amplifiers and oscillators is in how feedback is arranged, and where the poles of the transfer function lie in the complex plane. Oscillators have poles with a positive real part.

#### 6.3.1 Feedback amplifiers

This section is a rapid review of linear feedback theory. Every amplifier can be considered as an ideal, “open loop” amplifier with a feedback network connected to it. For simplicity we will assume we are discussing ideal voltage amplifiers with a voltage gain, and vanishing input admittance, output resistance and reverse current gain. The G matrix for such a device is evidently trivial:

\[
\begin{bmatrix}
  i_1 \\ v_2
\end{bmatrix} =
\begin{bmatrix}
  0 & 0 \\ G & 0
\end{bmatrix}
\begin{bmatrix}
  v_1 \\ i_2
\end{bmatrix}
\]  

(6)
Real amplifiers, of course, have a non-trivial \( G \) matrix; the non-vanishing matrix entries and finite source and load admittances we lump into a feedback amplifier, also with a trivial \( G \) matrix, of reverse gain \( H \):

![Feedback amplifier diagram]

**Figure 6-3: Feedback amplifier**

The gain of the combination of the two amplifiers is:

\[
A_y = \frac{G}{1-GH} \tag{7}
\]

If at any frequency \( GH=1 \) then the gain at that frequency is infinite, and there can be a signal at the output of the feedback amplifier with no input applied. Such an amplifier is called an oscillator, and the condition \( GH=1 \) (“the loop gain is unity”) is known as the oscillation condition. A more practical criterion is the Barkhausen criterion requiring that \( GH \) is real and \( GH>1 \). This ensures oscillations can start out of thermal noise.

Our next task is to show how a non trivial amplifier can satisfy the oscillation condition without explicit feedback connections. Let this amplifier be described by:

\[
\begin{bmatrix}
  i_1 \\
  v_2
\end{bmatrix}
= \begin{bmatrix}
  Y_{Hv} (\omega) & H_R (\omega) \\
  G(\omega) & Z_{\text{out}} (\omega)
\end{bmatrix}
\begin{bmatrix}
  v_1 \\
  i_2
\end{bmatrix} \tag{8}
\]

and be driven by a source impedance \( Z_s (\omega) \) and driving a load admittance \( Y_L (\omega) \).

The voltage gain of such an amplifier is:

\[
\frac{v_L}{v_1} = \frac{1}{1+Z_sY_{\text{in}}} \frac{1}{1+Z_{\text{out}}Y_L} \frac{G}{1+GH \frac{Y_LZ_s}{1+Z_sY_{\text{in}}}} \tag{9}
\]

i.e. the non-ideal amplifier behaves exactly like a feedback amplifier, one with

\[
H = -H \frac{Y_LZ_s}{1+Z_sY_{\text{in}}} \tag{10}
\]

And, of course the two voltage dividers at input and output. The effective scalar block diagram of the loaded amplifier driven by a finite Thevenin source is shown in **Figure 6-4**.
The inessential difference with the ideal feedback amplifier is that feedback is now implicit.

We have proven that any non-trivial amplifier is equivalent to a feedback amplifier, and therefore capable of oscillating, either by design or by accident.

There are many ways to get an oscillator out of amplifiers with positive feedback. An inverting amplifier with a 3rd or higher order filter in the feedback loop is a common one. Typically the feedback loop is designed to be second order, but there are always stray capacitances and inductances which raise the order of the filter. Other ways to synthesise an oscillator are shown in figure 3.2

Figure 6-4: Effective block diagram of fully loaded amplifier

\[
\frac{1}{1 + Z_s Y_{in}} \xrightarrow{G} \frac{1}{1 + Z_{out} Y_{L}} \\
\frac{H \frac{Y_L Z_s}{1 + Z_s Y_{in}}}{1 + Z_s Y_{in}}
\]

Figure 6-5: Three ways to make an oscillator. (a) voltage amplifier and series LC in the feedback path, (b) transconductor and parallel LC in the feedback path, (c) voltage amplifier and pure delay in the feedback path.
6.3.2 LC (tuned) oscillators

The Barkhausen condition can be simply satisfied with a tuned circuit, so that \( GH > 1 \) at some frequency. Common configurations using tuned resonators are the following:

1. The Colpitts oscillator

![Figure 6-6: Colpitts oscillator](image)

To analyse this circuit we note that the admittance looking into the node between \( C_1 \) and \( C_2 \) vanishes at a certain resonance frequency:

\[
Y_{c_1c_2} = \frac{j\omega \left( \frac{1}{C_1} + \frac{1}{C_2} - \omega^2 L \right)}{1 - \omega^2 LC_2} \Rightarrow Y_{c_1c_2} \left( \sqrt{\frac{C_1 + C_2}{LC_2}} \right) = 0
\]

(11)

with the base grounded. This suggests it is best to regard the Colpitts oscillator as an ideal, common base, current amplifier, with current gain \( h_i \approx 1 \). The current transfer ratio of the resonant circuit viewed as a current divider between \( C_2 \) and \( L \) is:

\[
\frac{i_{out}}{i_{in}} = \frac{j\omega C_2}{j\omega C_2 + \frac{1}{j\omega L}} = \frac{-\omega^2 LC_2}{1 - \omega^2 LC_2}
\]

(12)

At resonance the current gain of the feedback network is:

\[
\frac{i_{out}}{i_{in}} = \frac{-\omega^2 LC_2}{1 - \omega^2 LC_2} = \frac{C_1 + C_2}{C_2} = 1 + \frac{C_1}{C_2} > 1
\]

(13)

suggesting that there is more than unity real positive feedback and the Barkhausen criterion is satisfied.

One may be tempted to think that this is not the oscillation condition. However, all devices compress, i.e. their gain drops, when they amplify large signals. So a Colpitts oscillator will be excited by noise and will oscillate very near the resonant frequency we computed:

\[
f_0 = \frac{1}{2\pi} \sqrt{\frac{C_1 + C_2}{LC_2}}
\]

(14)

The oscillation amplitude will be determined by the active device compression.
A practical Colpitts oscillator may have the tank circuit connected to the VCC supply. It is not hard to show in this case that any series impedance on the VCC line will alter the resonant frequency, nor that the exact resonant frequency will be power supply dependent for any real transistor and any real biasing scheme. Both the noise injection and the power supply sensitivity present major design challenges to eliminate.

2. The Clapp Oscillator

The resonant frequency of a Colpitts oscillator can be altered by adding a series capacitor to the inductor. This raises the inductance value required for a given resonant frequency, and consequently the quality factor Q and the impedance level in the tank, lowering the current magnitude in the tank. A series tuned Colpitts oscillator is called the Clapp Oscillator. It is easy to show that the resonance frequency for the Clapp oscillator is:

\[ f_o = \frac{1}{2\pi} \sqrt{\frac{C_1 + C_2 + C_i}{LC_1C_2C_i}} \]  \hspace{1cm} (15)

This is a good opportunity to revise the concept of the quality factor of energy storage components and resonators, a concept that recurs in high frequency work. By definition, the Quality factor Q is the ratio of the maximum energy stored in the reactive elements to the energy dissipated in one cycle of the waveform. Thus, an inductor with some series resistive losses has a Q given by:

\[ Q = \frac{i_{\text{RMS}}^2 \omega L}{i_{\text{RMS}}^2 R} = \frac{\omega L}{R} \] \hspace{1cm} (16)

Note that this is just the argument of the complex impedance of the resistive inductor. Similarly, a leaky capacitor, shunted by a conductance G has a Q:

\[ Q = \frac{v_{\text{RMS}}^2 \omega C}{v_{\text{RMS}}^2 G} = \frac{\omega C}{G} \] \hspace{1cm} (17)

At resonance a lossy resonant circuit (a series RLC, for example) has a Q:
The quantity $Q = \frac{\omega_0}{L} = \frac{1}{\sqrt{LC}} = \frac{1}{RC}$ represents the ratio of the voltage to current amplitude of the resonator.

3. The Hartley oscillator

The Hartley oscillator is very similar to the Colpitts, using the dual of the tank circuit elements:

![Figure 6-8: A Hartley oscillator.](image)

Despite the difference in the tank circuit, this is still a shunt resonator presenting a null admittance at the resonant frequency:

$$f_0 = \frac{1}{2\pi \sqrt{\frac{1}{C(L_1 + L_2)}}}$$

The rest of the analysis is identical to that of the Colpitts oscillator.

4. Tuned amplifiers

These use a tuned load and explicit feedback, often involving a transformer. The Pierce Oscillator is an example of a transformerless tuned amplifier oscillator.

![Figure 6-9: A Pierce oscillator. The 3.3 mH inductor can be thought of as an infinite impedance (it IS a choke) and the crystal as a series RLC resonator.](image)

5. Crystal oscillators

A piezoelectric crystal exhibits a mechanical resonance. Near its resonance the crystal is electrically equivalent to an RLC resonator with very high quality factor. Crystals can be operated in series or parallel resonance mode, depending on the oscillator configuration they are used.
6. **Ring oscillators**

The oscillation condition can be met by providing a pure 1 cycle delay in the feedback path. Ring oscillators may use RC networks to generate such a delay, or multiple (inverting) amplifier stages to exploit the fact that an inversion is equivalent to half a cycle delay!

![Image of RC Phase Shift Oscillator]

Figure 6-10: Phase delay oscillator. The circuit oscillates at the frequency where the total delay over the RC network is one period.

7. **Relaxation Oscillators**

These use feedback to switch between two states of a current source, charging and discharging a timing capacitor. When a voltage level is reached the device changes state. The most popular relaxation oscillator device is the 555 timer which produces a square wave output. With the addition of an automatic gain control device a relaxation oscillator can be made to generate sine waves (Wien bridge oscillator).

![Image of Wien Bridge Oscillator]

Figure 6-11: The Wien bridge, a popular relaxation oscillator. Note the light bulbs (very high positive temperature resistance coefficient devices) used to balance the gain of the positive and negative feedback loops.
8. Voltage controlled oscillators

There are several ways to make a voltage controlled oscillator:

a. Use a varactor (i.e. a reverse biased diode) as the LC capacitor
b. Change the resonance of a crystal by “pulling” it, i.e. applying a voltage across it
c. Change the gain of the amplifier in a relaxation oscillator by altering the active element bias.

Figure 6-12: A varactor tuned Hartley Voltage controlled oscillator

6.4 Arithmetic with frequency

Two frequencies can be added or subtracted by multiplying the corresponding signals, and then filtering out the undesired mixing products. More often than not, the multiplication is a mask operation, i.e. the signal at $f_1$ is interrupted by a switch operated at the frequency $f_2$. However odd this technique may appear, it is correct, as it corresponds to the digital AND operator, which is indeed a multiplication operation. The frequency of a signal can be divided by an integer $N$ by using the signal to clock a counter.

In high frequency metrology we often need to multiply the frequency of a signal. Frequency multiplication by an integer is performed by exploiting the nonlinearity of electronic devices, such as diodes. Therefore, a frequency may be doubled by use of a diode, or tripled using a pair of diodes. Normal silicon diodes, and even Schottky diodes prove too slow for harmonic generation at large harmonic indices, losing about 7-10 dB of power per harmonic. (consider that the transit time of the diode makes it a low pass filter) When generation of high harmonic orders is required, a special type of diode, the step recovery diode is used. In the step-recovery diode the doping level is gradually decreased as the junction is approached. This reduces the switching time since the smaller amount of stored charge near the junction can be released more rapidly when changing from forward to reverse bias. The forward current can also be established more rapidly than in the ordinary junction diode.

Step recovery diodes can generate significant power up to harmonic indices of the order of 100. A device generating many harmonics out of an input waveform is called a comb generator. Comb generators are usually used in conjunction with very narrow ($Q >> 1000$) bandpass filters to select one of the harmonic lines generated. A common such filter uses a heated Yttrium Iron Garnet (YIG) crystal inside a magnetic field as a resonator. YIG filters are tuneable and have extremely high quality factors, in excess of $10^6$.

A second method to multiply the frequency of a signal employs phase locked loops and will be discussed later in these notes. The idea is that a feedback loop can be used to invert a mathematical
operation, hence, a feedback loop can be used to invert the frequency division implemented by a counter.

6.5 Frequency Measurement methods

6.5.1 Frequency-to-voltage conversion

![Frequency-to-voltage conversion diagram](image)

Figure 6-13: Frequency to voltage conversion

Frequency can be converted to a voltage by use of a Monostable Multivibrator. This is a device which emits a single pulse of fixed duration at a positive zero crossing event. The low pass filter is meant to reject the original AC filter, passing only the average of the output of the monostable. If $\tau$ is the monostable pulse duration, the average of the monostable output signal, of amplitude $V_0$ is:

$$\bar{V} = V_0 \frac{\tau}{T} = V_0 \tau f$$  \hspace{1cm} (20)

The uncertainty in the measurement is determined by the ripple which passes the averaging low pass filter, and hence decreases with the duration of the measurement. The range of the measurement is determined by the condition:

$$V_N < \bar{V} < V_0 - V_N$$  \hspace{1cm} (21)

with $V_N$ the RMS voltage noise of the measurement.
6.5.2 *Time Interval-to-Voltage Conversion*

The frequency to voltage conversion is a way to convert a time interval, the period, into a voltage. One can think of other analogue ways to perform this. For example, a positive ramp generator may be enabled by the positive edges of the limited (square) signal measured, and reset by the negative edges. The DC spectral component of the ramp generator of slew rate $S_+$, is, accounting also for the reset slew rate $S_-$ of the saw tooth generator:

$$V = \frac{TS_+}{8} \left( 1 + \frac{S_+}{S_-} \right)$$

The DC component is therefore proportional to the period. This type of instrument is limited at the low frequencies by the voltage swing of the ramp generator, and at the highest frequencies by the noise floor. Please note that the period to voltage measurement is sensitive on duty cycle, so that duty cycle adjustment to 50% (e.g. by a counter) is necessary for reliable measurements.

6.5.3 *Frequency counter*

The illustration of a frequency counter is shown below.

**Figure 6-15: Frequency counter**
This is a digital version of the analogue frequency to voltage converter of §3.5.1. The number of zero crossings in a fixed period of time is counted, and then normalised and displayed. The accumulated count, over a time interval $\Delta T=f_0/D$ (established by dividing a reference oscillator at frequency $f_0$ by a divider $D$) is:

$$N = \text{int}\left(\frac{\Delta T}{T}\right) = \text{int}\left(\frac{Df_0}{f_0}\right) \Rightarrow \frac{N}{D} < f < \frac{N+1}{D} \Rightarrow \lim_{D \to \infty} f = \frac{N+1/2}{D} f_0$$

(23)

Since the value of $N$ is uncertain by one unit, the accuracy of a frequency measurement depends on the duration of the measurement, and the range of the direct counting method has an obvious lower limit as $T$ approaches $\Delta T$, and a not so obvious high frequency limit, as the signal frequency approaches the speed limit (setup and hold time, and propagation delays) of the counter.

### 6.5.4 Period counter

![Period counter, including a prescaler](image)

The digital equivalent of the analogue period counting of §3.5.2 is to count, using a fixed reference frequency $F_0/D$, to the duration of one or more periods of the input signal. The count for $M$ periods will be:

$$N = \text{int}\left(MTf_0/D\right) \Rightarrow \frac{Mf_0}{D(N+1)} < f < \frac{Mf_0}{DN} \Rightarrow \lim_{M \to \infty} f = \frac{Mf_0}{D(N+1/2)}$$

(24)

A judicious use of frequency and period counting can provide a wide range of measurement.

### 6.5.5 Prescaled frequency measurement

In the previous sections we encountered both digital counters and (digital frequency dividers). A counter in the input signal path, which divides the input frequency by an integer $N$ is often called a prescaler. Prescalers are used to reduce the input frequency range to make it fit into the range of a modest instrument.

### 6.5.6 Heterodyne down conversion

If the frequencies of the input signal and the reference oscillators in figures Figure 6-15 and Figure 6-16 are similar, a beat frequency signal containing power in the sum and difference frequencies of the two waveforms will be generated. The beat frequency signal can then be filtered and fed into a low frequency instrument. For higher microwave frequencies, where the signal frequencies greatly
exceed the range of the instrument available, it may be impossible to construct a high enough frequency local oscillator to implement the reference oscillator. We then resort to the use of a comb generator and a YIG filter to multiply the LO frequency by an integer. By tuning the filter to different harmonics of the local oscillator we can have a very wide range for the instrument (Figure 6-17).

**Figure 6-17:** Heterodyne downconversion frequency counter, employing a comb generator to multiply the local oscillator frequency, and a YIG filter to select the correct harmonic

### 6.5.7 Harmonic downconversion

The techniques of downconversion and transfer oscillator can be combined in several conceivable ways. For example, the input signal may be acquired through a small ratio transfer oscillator (using a gate as a multiplier) and then measured with a heterodyne converter instrument. This technique is quoted as a low cost technique, since the requirements on both the harmonic generator and the microwave frequency counter are relaxed.

### 6.6 Voltage to frequency converters

The inverse of the frequency measurement process is also possible, allowing generation of a signal at a frequency dependent on the instantaneous amplitude of an electrical signal. The device is, of course, an FM modulator. Alternative names for FM modulators are Voltage Controlled Oscillators (VCO) and Current Controlled Oscillators (CCO) depending on whether the control signal is a voltage or current respectively.

#### 6.6.1 Definitions

A VCO has a **range** of frequencies it can generate, and a preferred frequency, the **free running frequency** when there is a null input applied. It also has a **gain** defined as:

\[
K_{\text{VCO}} = \frac{df}{dV}
\]  

(25)

Voltage controlled oscillators consist of a normal fixed frequency oscillator, with the resonant circuit's frequency (or the delay of a delay element) dependent on the input signal. In lower frequency applications one may use varactors to vary the tuning of an LC tank, or apply a bias on a crystal resonator to alter its resonance frequency. By changing the bias current of a transistor one may alter the delay in a feedback loop. At higher frequencies other specialised techniques may be used.
6.6.2 Phase noise and jitter

The output of any oscillator is noisy. The noise can be distinguished in **amplitude noise** and **phase noise**. For a sinusoidal signal the PSD's of amplitude and phase noise are each equal to half the total noise PSD of the signal. A perfect square wave only has phase noise, but then perfect square waves occur only at lower frequencies (say up to a few 100 MHz). A periodic signal is said to have **jitter**, defined as the RMS variation of the zero crossing time. Phase noise and jitter are extremely important in communications systems so they deserve a closer look.

Phase noise is random phase modulation of the oscillator output. Oscillator noise can be easily modelled by a linear feedback argument, originally due to David Leeson. We observe that an oscillator is really an amplifier with unity positive feedback. The feedback is arranged through a narrowband band pass filter, which in the proximity of the oscillator output can be described as a second order filter at $\omega_0$ with a quality factor (fractional bandwidth) $Q$.

In the steady state, the oscillation performs a mixing function between its output frequency and the power supply. Its output spectrum will therefore consist of the sum of 2 copies of the noise spectrum of the amplifier, which is in every case the sum of white and pink noise sources. Depending on whether the noise figure of the amplifier is low or high, the oscillator output spectrum has the form shown in Figure 6-18. Phase noise is measured in dBC, i.e. the percentage of the total power contained in a 1Hz bandwidth at a specified offset from the carrier. Typical numbers may be -90 dBC at 10kHz offset for a good integrated oscillator, or even -120 dBC at 10kHz for a good crystal oscillator.

We have already come across the concept of a frequency multiplier, which we will examine in more detail in the following pages. Evidently, since $\frac{df}{dt} \propto f$ the phase noise scales with the multiplication factor. This can have disastrous consequences, as large, of the order of $10^4 \approx 80$dB multiplication factors may be used in a communication system, greatly undermining the oscillator’s phase noise specification. So, the good crystal oscillator of the preceding example would only have 40dBC at 10kHz when multiplied by $10^4$!

![Figure 6-18: Noise spectrum of oscillators around their output frequency. The oscillator is assumed to consist of an amplifier in a positive feedback loop. (a) High noise figure amplifier (b) Low noise figure amplifier.](image-url)
6.7 Phase locked loops

6.7.1 Phase measurement

As we have seen, it is possible to measure the frequency of a signal. It is also possible to measure
the relative phase of two signals, as we now show.

Assume we have two sinusoidal signals at the same frequency, differing only by a constant phase
difference $\phi$. Then:

$$\sin(\omega t) \cdot \sin(\omega t + \phi) = \frac{1}{2} \left( \cos(\phi) - \cos(2\omega t + \phi) \right)$$  \hspace{1cm} (26)

So that by low pass filtering the output of a multiplier we obtain a signal proportional to the phase
difference between two periodic signals. There are many devices that can be used as phase
detectors, and we will discuss some of them.

6.7.2 Basic PLL structure

A phase locked loop (PLL) is a servo loop using a phase detector to control a voltage controlled
oscillator. A PLL is nonlinear, since the phase detector involves signal multiplication, not to
mention taking the logarithm of the time domain signal. However, under the assumption that the
loop is stable and sufficiently (to be quantified soon) close to its steady state, a linear analysis can
be applied. A block diagram of a PLL is shown in Figure 6-19.

![ PLL Block Diagram ]

**Figure 6-19:** A phase locked loop. Note that the VCO is an integrator.

The state variable in the loop is the phase of the periodic signal, and as a result the voltage
controlled oscillator, assumed to have a constant voltage-to-frequency gain is an integrator.
The gain of the frequency divider is $1/N$. The transfer function of the loop is:

$$B(s) = \frac{\phi_{out}}{\phi_{in}} = \frac{K_d K_o F(s)}{s + K_d K_o F(s) / N} = \frac{NKF(s)}{s + KF(s)}$$  \hspace{1cm} (27)

where $K = K_d K_o / N$

The PLL loop, in the steady state ($s=0$) has a phase gain $B=N$ and therefore multiplies an input
frequency by a factor of $N$.

The main difficulty in examining a PLL is conceptual, and due to the time appearing in two roles:
- As the independent variable for the oscillatory signals
- As the timescale of a disturbance on the control loop.
The frequency variable $s$, therefore, refers to the frequency of a phase perturbation (equivalently, it is the instantaneous baseband frequency of FM modulation) at the input of the loop (the $f_{\text{REF}}$ input).

A PLL can be used to generate a multiple (or a copy, if $N=1$) of a reference frequency at a definite phase relationship to it. If the input signal is perturbed in frequency (e.g. is an FM signal) the output will in general be in phase error. The error signal $\varepsilon_{\phi}(s)$ in response to a perturbation $\delta \phi(s)$ will be:

$$\varepsilon_{\phi}(s) = \left(1 - \frac{1}{N} B(s)\right) \delta \phi_i = \left(1 - \frac{KF(s)}{s + KF(s)}\right) \delta \phi_i = \frac{sF(s) \delta \phi_i(s)}{s + KF(s)}$$

(28)

As usual with control loops, a large loop gain leads to a small tracking error. However, the tracking error gets larger at large frequencies, or at large multiplying factors $N$.

### 6.7.3 Phase detectors

PLL loops are subject to a number of classification systems, depending on the type of signals (analogue or digital), the type of phase detector used and the order -or absence of- filter in the loop. An analog PLL uses a sinusoidal oscillator, and would not include a digital divider. A digital PLL uses a square wave VCO and also can include digital components in the loop.

A number of phase detectors, shown in Figure 6-20, are in common use.

An analog PLL will use an analog multiplier as a phase detector. Noticing that the current-voltage characteristic of a diode (or a BJT) is:

$$I_D = I_e e^{V/V_T}$$

(29)

it follows that if $V=V_1+V_2$ then $I=I_1I_2$. Then we can use diode or BJT loops to implement analog multipliers. This technique is known as the Translinear Principle and is examined in detail in advanced electronics courses. The simplest translinear amplifier is the differential amplifier, whose gain is proportional to the tail current. If a common emitter BJT is used to supply the tail current then the differential amplifier output is proportional to the product of the voltage driving the tail current transistor and the differential input voltage.

• Analog Signals: Balanced multiplier

$$V_{out} = A \cos \omega_1 t \cdot B \cos(\omega_2 t + \varphi) = \frac{AB}{2} \left( \cos \left( \omega_2 - \omega_1 \right) t + \varphi \right) + \cos \left( \omega_2 + \omega_1 \right) t + \varphi$$

• Digital Signals:

6.7.3.1 Type 1 = Exclusive-OR

If the signal and the VCO are at the same frequency $f_0$ and differing in phase by $\varphi$, the output signal will be at a frequency $2f_0$ and will have a duty cycle $D = \frac{\varphi}{\pi}$ for $0 < \varphi < \pi$ and $D = 2 - \frac{\varphi}{\pi}$ for $\pi < \varphi < 2\pi$. The output signal has a DC component as shown in Figure 6-20. To maximize lock range the type 1 detector requires an equal 50% duty cycle on both its input signals. A PLL with type 1 phase detector will lock at a phase difference of $\pi/2$. The XOR gate as a phase detector, despite its simplicity is not very popular, due to its small linear range. However, we will restrict the discussion to this type of detector.
The range of a phase detector can be doubled if it can detect which signal is leading in phase. The following 2 types of phase detector do just this, and have therefore double the range of the type 1 detector. Furthermore, since the input signals are not equivalent, the centre of the linear region of these detectors is when the signals are in phase.

6.7.3.2 Type 2: Positive-edge triggered J-K flip-flop.
This detector fires off at leading and trailing edges of the waveforms. It turns out that the duty cycle of the signals is unimportant. This detector has a phase difference range of 2\pi. When the two signals are in lock the detector output is at a frequency equal to the signal frequency. This ripple is contributes to phase noise in the output of the PLL.

6.7.3.3 Type 3: Positive-edge triggered ‘tri-state’ detector
The advantage of this detector is that when the inputs are in lock there is no ripple signal. The range of this detector is between –2\pi and +2\pi phase difference. This is probably the most commonly used detector.

The gain of digital detectors is easily computed from their linear range and the logic levels between which they are operated: 

\[ K_d = \frac{V_{hi} - V_{lo}}{\delta \theta} \]

Note that the definitions of type2 and type 3 detectors may be reversed in some references.
<table>
<thead>
<tr>
<th>Input signals</th>
<th>Circuit</th>
<th>$V_{out} = f(\theta)$</th>
</tr>
</thead>
</table>
| \( v_1, v_2 \) | Four-quadrant multiplier | \[
\begin{align*}
\theta &= 0, \pi/2, \pi, 3\pi/2 \\
V_{out} &= V_0 \\
\end{align*}
\] |
| \( v_1, v_2 \) | Exclusive OR | \[
\begin{align*}
\theta &= 0, \pi/2, \pi, 3\pi/2 \\
V_{out} &= V_0 \\
\end{align*}
\] |
| \( v_1, v_2 \) | JK master/slave FF | \[
\begin{align*}
\theta &= 0, \pi/2, \pi, 3\pi/2 \\
V_{out} &= V_0 \\
\end{align*}
\] |

**Figure 6-20:** Common Phase/frequency detectors (From U. Rohde, *Microwave and Wireless Synthesizers*)
6.7.4 First order loops

6.7.4.1 Frequency response
The loop filter is not necessary for the operation of the PLL. When there is no filter in the loop, the transfer function eq. (27) reduces to

\[ B(s) = \frac{\phi_{out}}{\phi_{in}} = \frac{NK}{s + K} \] (30)

And the loop is called a first order loop. First order loops are commonly constructed with type 1 detectors.

6.7.4.2 Tracking
The tracking characteristics of a loop can be evaluated by examining the loop’s response to a phase disturbance of the form:

\[ \theta(t) = at^n \] (31)

The case \( n=0 \) corresponds to a step to the input phase; \( n=1 \) is a step change in frequency, and \( n=2 \) a frequency ramp.

The steady state phase error is (by the final value theorem):

\[ \delta\phi = \lim_{s \to \infty} s\phi(s) = \lim_{s \to 0} s\phi(s) \] (32)

and

\[ \frac{\phi(s)}{\phi_i(s)} = 1 - \frac{1}{N} B(s) = 1 - \frac{K}{s + K} = \frac{s}{s + K} \] (33)

then,

\[ \delta\phi_{SS} = \lim_{s \to 0} s\phi(s) = \lim_{s \to 0} s\theta(s) = \lim_{s \to 0} \frac{as^{1-n}}{s + K} \] (34)

since the Laplace transform of the input phase disturbance is: \( L\theta(t) = \frac{an!}{s^{n+1}} \)

Equation (34) suggests that the first order loop has a zero steady state phase error for an input step phase change. If a frequency step is applied, the loop will follow the change with a steady state error \( \delta\phi = aN / KdK_0 \), while the steady state phase error to a frequency ramp (\( n=2 \)) is infinite, i.e. the PLL fails to track a frequency ramp.

6.7.4.3 Bandwidth and risetime
The bandwidth of the 1st order loop, i.e. the frequencies of phase modulation it will track is \( K = K_d K_0 / N \). The risetime, i.e. the time the loop takes to track a change of phase at its input from 10% to 90% of the transient is: \( \delta t = 2.2/B = 2.2/K \)

6.7.4.4 Capture range and acquisition time
Assume the loop is not locked. In this case, the VCO is running at its free running frequency. It is useful to know the range of frequencies, relative to its free running frequency, the PLL will capture, i.e. it will lock on. This, for a type 1 loop turns out to be

\[ \Delta\omega_{\text{capture}} = \frac{K_d K_0}{N} = K \] (35)

The acquisition time, with a Type 1 detector is approximately:
where \( \theta_f \) refers to the final phase error. A large gain helps fast acquisition, as well as provides a wide capture range.

### 6.7.5 Second order loops – Lowpass filter

#### 6.7.5.1 Frequency response

The simplest loop filter we can introduce is a single pole low pass filter. This we usually write as:

\[
F(s) = \frac{1}{1 + \tau s}
\]

The loop transfer function now becomes:

\[
B(s) = \frac{NKF(s)}{s + KF(s)} = \frac{NK}{s(1 + \tau s) + K} = \frac{N}{(s / \omega_n)^2 + 2\zeta (s / \omega_n) + 1}
\]

Which is a second order low pass function with

\[\omega_n = \sqrt{\frac{K}{\tau}} , \quad 2\zeta = \frac{1}{Q} = \frac{\omega_n}{K} = \sqrt{\frac{1}{\tau K}}\]

the magnitude of the frequency response is then:

\[
|B(s)| = \frac{\phi_o}{\phi_i}(\omega) = \frac{N}{\sqrt{(1 - \omega^2 / \omega_n^2)^2 + (2\zeta \omega / \omega_n)^2}}
\]

which peaks at a frequency (of disturbance!)

\[\omega_p = \omega_n \sqrt{1 - 2\zeta^2}\]

and has a maximum:

\[M_p = \max(|B(\omega)|) = \frac{1}{2\zeta \sqrt{1 - \zeta^2}}\]

The phase response of the loop is:

\[
\arg\left(\frac{\phi_o}{\phi_i}(\omega)\right) = \arctan\left(\frac{2\zeta \omega}{\omega_n \left(1 - \omega^2 / \omega_n^2\right)}\right)
\]

#### 6.7.5.2 Bandwidth and rise time

The loop has a 3dB bandwidth around this maximum is:

\[B_n \cong \zeta \omega_n \sqrt{1 - \zeta^2} / \sqrt{1 - 2\zeta^2} , \quad \lim_{\zeta \to 0} \delta \omega = \zeta \omega_n\]
and once again an approximate rise time may be defined as $\delta t = \frac{2.2}{B} = \frac{2.2}{\zeta \omega_n}$.

The biggest drawback of the simple low pass second order loop is that the natural frequency $\omega_n$ and the damping factor $\zeta$ are intimately linked. Furthermore, the damping factor is inversely proportional to the loop gain, which we require large in order to have fast acquisition. We therefore require independent control of the loop bandwidth and the damping factor. This is done with a lead-lag filter (second type analog filter in fig 4.10). This is the simplest realistic (and useful) loop description.

6.7.5.3 Tracking
The steady state tracking error, to phase disturbances $\theta(t) = at^n$

$$\lim_{s \to 0} \epsilon_p(t) = \cdots = \lim_{s \to 0} \frac{s^2 \theta(s)}{s + KF(s)} = \lim_{s \to 0} \frac{as^{2-n}}{KF(s)} = \lim_{s \to 0} \frac{as^{2-n}(1 + \tau s)}{K}$$

(45)

and therefore the discussion of the 1st order loop applies. This loop will track step phase changes and step frequency changes with zero steady state error, and linear frequency changes with a finite steady state phase error. Equation (45) gives us a hint on how we can construct a PLL that will track with no phase error frequency ramps: The filter must have one or more poles at zero. For this reason active filters with capacitive feedback are often used, as shown in figure 4.10.

The existence of a loop natural frequency has an interesting consequence. Despite the conclusions of a the simplistic analysis above, this loop will now track changes of the input reference frequency as long as the rate of change of reference frequency satisfies:

$$\frac{d(\Delta \omega)}{dt} < \omega_n^2$$

(46)

and the maximum rate the VCO can be swept to achieve lock when initially not in lock is just half the value above.

The capture range and the hold ranges are both equal to $K$ rad/sec. Finally, the acquisition time, to acquire a difference $\Delta f$ from the free running frequency is:

$$\Delta \tau \cong \frac{4(\Delta f)^2}{B_n^3}$$

(47)

6.7.5.4 Second order loop with lead lag filter
We can write the transfer function of a lead-lag filter as:

$$F(s) = \frac{1 + \tau_s s}{1 + \tau_1 s}$$

(48)

With this filter, the PLL loop transfer function becomes:

$$B(s) = N \frac{K(1 + \tau_s s)}{s(1 + \tau_s s) + K(1 + \tau_2 s)} = N \frac{s\omega_n (2\zeta - \omega_n/K) + \omega_n^2}{s^2 + 2s\zeta \omega_n + \omega_n^2}$$

(49)
with the natural frequency and damping factor given by:

$$\omega_n = \sqrt{\frac{K}{\tau_1}}, \quad \zeta = \frac{1}{2} \sqrt{\frac{1}{\tau_1 K} (1 + \tau_2 K)} \tag{50}$$

The inclusion of the zero in the filter transfer function gives us control of the capture range. This is now:

$$\Delta \omega_c = K \frac{\tau_1}{\tau_2} \tag{51}$$

It can be shown that the acquisition time of the loops considered so far is exceedingly long, due to the residual phase error we discussed, but also the small linear range of the type 1 detector. Better performance can be obtained by use of a tri-state detector, and a lead-lag filter with a pole at zero.

<table>
<thead>
<tr>
<th>Type</th>
<th>Passive</th>
<th>Active</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td><img src="image1.png" alt="Circuit 1" /></td>
<td><img src="image4.png" alt="Circuit 4" /></td>
</tr>
<tr>
<td>2</td>
<td><img src="image2.png" alt="Circuit 2" /></td>
<td><img src="image5.png" alt="Circuit 5" /></td>
</tr>
<tr>
<td>3</td>
<td><img src="image3.png" alt="Circuit 3" /></td>
<td><img src="image6.png" alt="Circuit 6" /></td>
</tr>
</tbody>
</table>

**Figure 6-21**: Common loop filters for 2\textsuperscript{nd} order PLLs. (From U. Rohde, *Microwave and Wireless Synthesizer*).

### 6.8 Some PLL applications

#### 6.8.1 Fractional PLL

Modern communications applications require the accurate generation of many frequencies closely spaced together, representing the carriers for the different channels in the system. Typical requirements may be 30kHz channel spacing around carriers of 1GHz. Such frequencies can indeed be generated by using a reference frequency equal to the channel spacing, and large (of the order of 30000) multiplication factors. We have already mentioned the detrimental effect on phase noise this may have. Large multiplication factors also slow down acquisition since the multiplication factor divides the loop gain.

An alternative method exists to generate frequencies at a spacing much smaller than the reference frequency. This is called fractional synthesis, and is extremely popular in consumer telecommunications systems. It has even started appearing in the laboratory in cheap (but high performance) bench-top signal generators.
The idea is simple, as explained in Figure 6-22. In the course of every $M$ cycles of the baseband frequency the divider spends $K$ cycles dividing by $N+1$ and another $M-K$ cycles dividing by $N$. As a result, the output frequency, if suitably filtered, is centred around the average:

$$f_0 = f_{\text{REF}} \left( \frac{K(N+1)+(M-K)N}{M} \right) = f_{\text{REF}} \left( \frac{N+K}{M} \right)$$

(52)

Evidently, the output frequency spacing is now $f_{\text{REF}} / M$. Although we may be tempted to think we gained 20dB/decade of $M$ in phase noise it turns out the gain is closer to 10dB/decade. Still, this may represent 30-40dB in phase noise for a typical synthesiser.

The dual modulus controller is another popular circuit, namely the Delta-Sigma Analog to Digital converter we will discuss later in the course.

**Figure 6-22: Fractional PLL synthesiser**

### 6.8.2 Transfer Oscillator

A very interesting (and useful!) measurement technique exploits a PLL loop and a comb generator to generate a signal at a fraction (a sub-harmonic) of the input signal's frequency in phase lock to it. The lower frequency signal is then conventionally measured. The principle is shown in Figure 6-23. The multiplication factor $N$ in the frequency multiplication is determined by successively tuning the YIG filter to different $N$ and noting the difference in readings. A YIG (Yttrium Iron Garnet) filter is a magnetic resonator whose frequency depends on the ambient magnetic field generated by a small electromagnet. It can be tuned over several decades at GHz frequencies and is extremely narrowband with $Q \approx 10^6$.

**Figure 6-23: Frequency measurement using a transfer oscillator to scale the frequency of the input signal down to the range of the available instrument.**