Propositional Logic

• This lecture marks the start of a new section of the course.
• In the last few lectures, we have had to reason formally about concepts.
• This lecture introduces the mathematical language to express this reasoning: the language of logic.
• This lecture will cover
  – What is logic?
  – Propositions
  – Implications
  – Logical equivalence
  – Logic and the English Language

What is logic?

• This is a contentious question! We will play it safe, and stick to:
  – “The systematic use of symbolic techniques and mathematical methods to determine the forms of valid deductive argument.” [Shorter Oxford Dictionary].
• Essentially, logic formalizes our reasoning process.
  – It provides a common language through which we can demonstrate to each other that our reasoning is valid.
• All the proofs of the theorems we have seen in Lectures 1 – 3 are based on logic.

Propositions

• Definition
  – A proposition is a declarative sentence that is either true or false, but not both.
• Examples
  – All ISE students wear glasses. [false]
  – This lecture course is given at Imperial College. [true]
  – If a relation R is transitive, then R^n is transitive. [true]
  – Hopefully this lecture will finish on time. [NOT a proposition]
• We will often use symbols to denote propositions
  – Let p be the proposition “all ISE students wear glasses”.
• Propositional logic is the branch of logic that deals with reasoning about propositions.

Negation

• What can we do with propositions?
• We can use them as building blocks to construct further propositions, called compound propositions.
  – “all ISE students wear glasses” is related to the proposition “not all ISE students wear glasses”, through negation.
• Definition
  – Let p be a proposition. The statement “it is not the case that p” is another proposition, called the negation of p.
• Notation
  – The negation of p is denoted ¬p (read as “not p”).
• Example
  – Let p be the proposition “this lecture course is given at Imperial College”. Then ¬p is the proposition “it is not the case that this lecture course is given at Imperial College”, or, more simply, “this lecture course is not given at Imperial College”.
Conjunction and Disjunction

- **Definition**
  - Let $p$ and $q$ be propositions. The proposition “$p$ and $q$”, denoted $p \land q$, is the proposition that is true when both $p$ and $q$ are true, and false otherwise. $p \land q$ is called the conjunction of $p$ and $q$.

- **Definition**
  - Let $p$ and $q$ be propositions. The proposition “$p$ or $q$”, denoted $p \lor q$, is the proposition that is false when both $p$ and $q$ are false, and true otherwise. $p \lor q$ is called the disjunction of $p$ and $q$.

- **Examples**
  - If $p$ is the proposition “ISE students love logic”, and $q$ is the proposition “ISE students are crazy”, then
    - $p \land q$ is the proposition “ISE students love logic and are crazy”
    - $p \lor q$ is the proposition “ISE students either love logic, or are crazy, or both”

- **Note** the syntax is different to that used in digital electronics. This is standard logical syntax – please stick to it!

Implications

- **Definition**
  - Let $p$ and $q$ be propositions. The proposition “$p$ implies $q$”, denoted $p \rightarrow q$, is the proposition that is false when $p$ is true and $q$ is false, and true otherwise. $p \rightarrow q$ is called the premise or hypothesis, and $q$ is called the conclusion or consequence.

- **Examples**
  - Let $p$ be the proposition “I revised”, and let $q$ be the proposition “I passed”. Then $p \rightarrow q$ may be expressed as:
    - “if I revised, I passed”;
    - “I passed if I revised”;
    - “I revised only if I passed”.

  - **Note** that the premise and conclusion need not have any relationship.
    - “if all ISE students wear glasses, then $2 + 3 = 5$.”
      - This proposition is TRUE (it is irrelevant whether all ISE students wear glasses, because $2 + 3 = 5$).
    - “if all ISE students wear glasses, then $2 + 3 = 6$.”
      - This proposition is TRUE (because not all ISE students wear glasses).
    - “if we are at Imperial College, then $2 + 3 = 6$.”
      - This proposition is FALSE (because we are at Imperial College, but $2 + 3 \neq 6$).

  - **This use of the term “implies” is distinct from standard English usage.**
    - In English “implies” can mean “causes”.
    - English considers the meaning of the propositions.
    - Logic only considers whether they are true or false.

Converse, Contrapositive, and Inverse

- **The meaning of “iff” follows directly**
  - $a$ iff $b$ is the same as $(a \rightarrow b) \land (b \rightarrow a)$, and is usually written $a \leftrightarrow b$.
  - So $a \leftrightarrow b$ is true either when $a$ and $b$ are both true, or when $a$ and $b$ are both false (and is otherwise false).

- **Definition**
  - The converse of a proposition $a \rightarrow b$ is the proposition $b \rightarrow a$.
  - The contrapositive of an proposition $a \rightarrow b$ is the proposition $\neg b \rightarrow \neg a$.
  - The inverse of a proposition $a \rightarrow b$ is the proposition $\neg a \rightarrow \neg b$. 
Logical Equivalence

- **Definition**
  - A *tautology* is a compound proposition that is always true, no matter what the truth values of the propositions that occur in it.
  - A *contradiction* is a compound proposition that is always false, no matter what the truth values of the propositions that occur in it.
  - A *contingency* is a proposition that is neither a tautology nor a contradiction.

- **Examples**
  - \( p \lor \neg p \) is a tautology (either \( p \) is true, or \( \neg p \) is true).
  - \( p \land \neg p \) is a contradiction (\( p \) cannot be both true and false).

- **Definition**
  - The propositions \( p \) and \( q \) are logically equivalent, denoted \( p \equiv q \), if \( p \iff q \) is a tautology.

- **Example**
  \[ \neg(p \lor q) \equiv \neg p \land \neg q. \]
  You know this as De Morgan’s Theorem.

- **We can express an implication in terms of a disjunction and a negation**
  \[ p \to q \equiv \neg p \lor q. \]

- **It therefore follows that an implication is logically equivalent to its contrapositive.**
  \[ p \to q \equiv \neg q \to \neg p. \]
  - “If I revised, I passed” = “If I didn’t pass, I didn’t revise”.

- **By the same reasoning, its converse is logically equivalent to its inverse.**
  \[ q \to p \equiv \neg p \lor q \equiv \neg q \to \neg p. \]

- **The most common mistake in logic is think an implication is logically equivalent to its inverse**
  - “If I revised, I passed” is *not* equivalent to
  - “If I didn’t revise, I didn’t pass”

- **Operator Precedence**
  - From your digital electronics, you are familiar with some rules of logical precedence.
    \[ a \land \neg b = a \land (\neg b) \quad \text{(apply the “not” first)} \]
    \[ a \lor b \land c = a \lor (b \land c) \quad \text{(apply the “and” before the “or”)}. \]

- **To these rules, we need to add the rule for implications:** these are given lowest precedence.
  \[ a \lor b \to c = (a \lor b) \to c \quad \text{(implications last)} \]

- **As usual, add brackets as required.**
Logic and the English Language

• Translating English to logic is not always easy.
• Try to identify the basic propositions, and use the logical operators to build them into a compound proposition equivalent to the original English.
• Examples:\n  "I bought beans and peas" = "I bought beans" \(\wedge\) "I bought peas".
  "I am not good at logic" = \(\neg\)("I am good at logic").
  "If it's raining then it's not sunny" = raining \(\rightarrow\) \(\neg\)sunny.
• But English is an inexact language (that's why we need symbolic logic!)
  "You can have chicken or fish" = "you can have chicken" \(\wedge\) "you can have fish" (note the English "or" here is a logical "and"!)

Examples:\n1) "I bought beans and peas" = "I bought beans" \(\wedge\) "I bought peas".
2) "I am not good at logic" = \(\neg\)("I am good at logic").
3) "If it's raining then it's not sunny" = raining \(\rightarrow\) \(\neg\)sunny.

Test Your Knowledge

• True or false?
  a) \(2 + 2 = 4\) \(\leftrightarrow\) \(1 + 1 = 2\)
  b) \(1 + 1 = 2\) \(\leftrightarrow\) \(2 + 3 = 6\)
  c) \(1 + 1 = 3\) \(\leftrightarrow\) \(2 + 3 = 6\)
  d) if \(1 + 1 = 2\), then \(2 + 2 = 5\)
  e) if pigs can fly, then \(1 + 1 = 3\).

• Express in English the (i) contrapositive, (ii) inverse, of each of these statements.
  a) "if it snows today, I will ski tomorrow."
  b) "I come to College whenever George gives me a quiz."
  c) "A positive integer is prime if it has no divisors other than itself and 1."

Summary

• This lecture has covered
  – What is logic?
  – Propositions
  – Implications
  – Logical equivalence
  – Logic and the English Language
• We are still unable to break down some propositions in terms of simpler ones
  – “every slave has a master” (how do we deal with every?)
• Next lecture will look at predicate logic, which will solve this problem.
Predicate Logic

• In the previous lecture, we examined propositional logic.
• In this lecture, we will extend this to predicate logic.
• The lecture will cover
  – why predicate logic?
  – what is a predicate?
  – the universal and existential quantifiers
  – binding variables
  – nested quantification
  – predicate logic and the English language

Why Predicate Logic?

• Propositional logic has some limitations
  – How do we express
    “all ISE students are clever”, in terms of the cleverness (or otherwise) of particular ISE students?
    “there is an ISE student wearing red socks”, in terms of whether each individual ISE students is wearing red socks?
  – Ideas like this become particularly important when trying to prove theorems
    say we want to show “all real numbers except zero have a real reciprocal which, when the two are multiplied, gives 1”.
    we need to be able to express this idea formally.

What is a predicate?

• We will generalize the concept of a proposition to a predicate.
  – A predicate takes one or more variables as arguments.
• Example
  – Let P(x) denote the statement “x > 3”. Then P(1) is false, P(4) is true.
  – Let P(x) denote the statement “x is a Lecturer in EEE at Imperial College”. Then P(George) is true, but P(Amanda) is false.
• P(x) is neither true nor false until we specify a value for x.

The Universal Quantifier

• Let P(x) be the predicate “x has a heart”.
  – We can express that some particular humans have a heart:
    P(George), P(Steve), P(Raj).
  – How can we say “all humans have a heart?”
• Predicates start to help us express concepts like this by using quantification.
• Definition
  – The universal quantification of P(x) is the proposition “P(x) is true for all values of x in the universe of discourse”.
• The term “universe of discourse” simply means “all x we wish to discuss”.
• Example
  – With the predicate above, we would say that for the universe of discourse consisting of humans, the universal quantification of P(x) is true.
The Universal Quantifier

- **Notation**
  - We will write $\forall x \ P(x)$ (read as “for all $x$, $P(x)$”) for the universal quantification of $P(x)$.
- **The universe of discourse must, in general, be stated when constructing a quantification.**
  - Let $P(x)$ be the predicate “$x^2 \geq 0$”.
  - $\forall x \ P(x)$ is true if the real numbers are the universe of discourse, but false if the complex numbers are the universe of discourse.
- **Often we can state the universe of discourse concisely using set notation**
  - $\forall x \in \mathbb{R}, P(x)$ “for all real $x$, $P(x)$”
  - $\neg(\forall x \in \mathbb{C}, P(x))$ “it is not the case that for all complex $x$, $P(x)$”

[both these propositions are true if $P(x)$ is the predicate “$x^2 \geq 0$”]

The Existential Quantifier

- **Let $P(x)$ be the predicate “$x$ is wearing something red”**.
  - We can express the propositions “George is wearing something red”, $P(\text{George})$, and “Peter is wearing something red”, $P(\text{Peter})$.
  - How could we say “there is an ISE student wearing something red”?
- **Definition**
  - The existential quantification of $P(x)$ is the proposition “There exists an element $x$ in the universe of discourse such that $P(x)$ is true”.
- **Notation**
  - We will write $\exists x \ P(x)$ (read as “there exists an $x$ such that $P(x)$”) for the existential quantification of $P(x)$.
- **Example**
  - Let $J$ be the set of ISE students. We can now construct the proposition as $\exists x \in J, P(x)$.

Binding Variables

- In general, a predicate may have several variables.
  - Let $P(x,y)$ be the predicate “$x^2 > y$”.
- **If we fix the value of a variable, or use a quantifier on it, we say the variable is bound.**
  - A variable that is not bound is called free.
- **Example**
  - For the predicate above, $P(x,y)$ has two free variables.
  - $P(1,y)$ has one free variable.
  - $\forall y \ P(1,y)$ has no free variables.
- **All variables must be bound to turn the predicate into a proposition.**
- Just like in programming, binding variables have a scope.
Relationships

• We may wish to negate a quantified expression.
  Let $P(x)$ be the predicate “$x$ likes logic”.
  $\neg(\forall x \in J \ P(x)) \equiv \text{"not all ISE students like logic"}$
  Note that this is the same as “there is an ISE student who
  doesn’t like logic”:
  $\neg(\forall x \in J \ P(x)) \equiv \exists x \in J \ \neg P(x)$.

• In general,
  $\neg \forall x \ P(x) \equiv \exists x \ \neg P(x)$.
  $\neg \exists x \ P(x) \equiv \forall x \ \neg P(x)$.

• You may notice this as De Morgan’s theorem in
  disguise.

Nested Quantification

• When a predicate has more than one variable, it is
  possible to nest quantifiers.
  – A nested quantifier is one that occurs within the scope of
    at least one other quantifier.

• Examples
  – $\forall x \exists y \ (x + y = 0)$. “For all $x$, there is a $y$ such that
    $x + y = 0$”. Here $\exists y$ is a nested quantifier. This proposition
    is true (if the real numbers are the universe of discourse).
  – $\exists y \forall x \ (x + y = 0)$. “There is a $y$ such that for all $x$,
    $x + y = 0$”. Here $\forall x$ is a nested quantifier. This
    proposition is false (there is no $y$ such that adding any
    number to it makes zero).

Nested Quantification

• Further example.
  – Let $C(x)$ be the statement “$x$ has a computer”, $F(x,y)$ be
    the statement “$x$ and $y$ are friends”, and the universe of
    discourse be the set of ISE students.
  – What does $\forall x(\ C(x) \lor \exists y(C(y) \land F(x,y)) \ )$ mean?
    “for all $x$, either $x$ has a computer, or there is a $y$ such
    that $y$ owns a computer and $x$ is friends with $y$, or both”.
  – More simply: “all ISE students either own a computer,
    have a friend in ISE who owns a computer, or both”.

Quantification of Two Variables

• To clarify the rules, let us look at all four combinations of the
  quantification of two variables.

<table>
<thead>
<tr>
<th>Proposition</th>
<th>When True</th>
<th>When False</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\forall x \ \forall y \ P(x,y)$ or</td>
<td>$P(x,y)$ is true for every pair $x$, $y$.</td>
<td>There is a pair $x$, $y$, for which $P(x,y)$ is false.</td>
</tr>
<tr>
<td>$\forall y \ \forall x \ P(x,y)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\forall x \ \exists y \ P(x,y)$</td>
<td>For every $x$ there is a $y$ for which $P(x,y)$ is true.</td>
<td>There is an $x$ such that $P(x,y)$ is false for every $y$.</td>
</tr>
<tr>
<td>$\exists x \ \forall y \ P(x,y)$</td>
<td>There is an $x$ for which $P(x,y)$ is true for every $y$.</td>
<td>For every $x$ there is a $y$ for which $P(x,y)$ is false.</td>
</tr>
<tr>
<td>$\exists y \ \forall x \ P(x,y)$</td>
<td>There is a pair $x$, $y$ for which $P(x,y)$ is true.</td>
<td>$P(x,y)$ is false for every pair $x$, $y$.</td>
</tr>
<tr>
<td>$\exists x \ \exists y \ P(x,y)$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Predicate Logic and English

• Example 1
  – Let F(x) be the predicate “x is female”.
  – Let P(x) be the predicate “x is a parent”.
  – Let M(x,y) be the predicate “x is the mother of y”.
  – Let the universe of discourse be the set of all people.
  – We can express the statement “If a person is female and is a parent, then this person is someone’s mother”.
    • “if …, then …” requires the use of implication.
    • “is someone’s mother” requires the use of existential quantification.
    • “a person” means this statement is true for all people – universal quantification.
      \[ \forall x \ ( F(x) \land P(x) \rightarrow \exists y \ M(x,y) ) \]

Predicate Logic and English

• Example 2
  – “The sum of two positive integers is positive”.
    • The hidden word here is “any”, i.e. the sum of any two positive integers is positive.
    • There is an implication: if integer1 is positive and integer2 is positive, then integer1+integer2 is positive.
  – Let us define the universe of discourse as the set of integers.
    \[ \forall x \ \forall y \ ( (x > 0) \land (y > 0) \rightarrow (x+y > 0) ) . \]

Test Your Knowledge

• True or false? (the universe of discourse is the set of real numbers)
  a) \[ \forall x \ \exists y \ ( x^2 = y ) \]
  b) \[ \exists x \ \forall y \ ( xy = 0 ) \]
  c) \[ \forall x \ ( x=0 \rightarrow \exists y ( xy=1 ) ) \]
  d) \[ \exists x \ \forall y \ ( y\neq 0 \rightarrow xy=1 ) \]

• Let L(x,y) be the statement “x loves y”, and let the universe of discourse is the set of all people in the world. Express the following using quantifiers:
  a) Everybody loves Jerry.
  b) Everybody loves somebody.
  c) There is somebody whom everybody loves.
  d) Nobody loves everybody.

Summary

• The lecture has covered
  – why predicate logic?
  – what is a predicate?
  – the universal and existential quantifiers
  – binding variables
  – nested quantification
  – predicate logic and the English language
• We now have a good grounding in basic logic.
• Next lecture will apply our knowledge to help us understand standard proof techniques.
Methods of Proof

• We have now introduced the fundamental concepts of logic.
• In this lecture, we will put them to work by examining methods of proof.
• This lecture will cover
  – What is a theorem, lemma, and proof?
  – Valid arguments.
  – Inference for quantified statements.
  – Common errors.

Conjectures, Theorems, and Proofs

• Definition
  – A conjecture is a statement that is neither known to be true nor known to be false.
• Definition
  – A theorem is a statement that can be shown to be true.
• Definition
  – A lemma is a minor theorem, usually used in the proof of another theorem.
• Definition
  – A proof of a theorem is a sequence of logically connected statements, forming an argument for the truth of the theorem.
  • A proof begins with the assumptions (hypotheses, axioms), and ends with the theorem.

Rules of Inference

• To logically obtain a statement from one (or more) other statements, we need a rule of inference.
  – rules of inference describe formally valid logical reasoning.
• Aristotle (384-322 BC) first formalised a rule of inference exemplified by this series of statements:
  – All men are mortal.
  – Socrates is a man.
  – Therefore, Socrates is mortal.
• The idea of a rule of inference is to abstract away the details, and concentrate on the form of the argument.
• Each rule of inference therefore corresponds to a tautology
  – the rule must be true, no matter what the truth of the statements.
  – e.g. Aristotle’s argument is true, whether Socrates is a man or not.

Modus Ponens and Addition

• Definition
  – Modus ponens is the rule of inference based on the tautology \( p \land (p \rightarrow q) \rightarrow q \).
• Example
  – “If it is sunny and I play football whenever it’s sunny, then I play football.”
• Definition
  – Addition is the rule of inference based on the tautology \( p \rightarrow p \lor q \).
• Example
  – “If \( x < 0 \), then \( x < 0 \) or \( x = 0 \).”
Simplification and Modus Tollens

- **Definition**
  - *Simplification* is the rule of inference based on the tautology \( p \land q \rightarrow p \).

- **Example**
  - “If this lecture finishes on time and I am wearing red socks, then this lecture finishes on time.”

- **Definition**
  - *Modus tollens* is the rule of inference based on the tautology \( \neg q \land (p \rightarrow q) \rightarrow \neg p \).

- **Example**
  - “If I don’t play football and I play football whenever it is sunny, then it isn’t sunny.”

The Hypothetical Syllogism

- **Definition**
  - The *hypothetical syllogism* is the rule of inference based on the tautology \( (p \rightarrow q) \land (q \rightarrow r) \rightarrow (p \rightarrow r) \).

- **Example**
  - “If Avril smiles when she does logic, and Avril is happy when she smiles, then Avril is happy when doing logic.”
  - Here \( p \) is the proposition “Avril does logic”, \( q \) is the proposition “Avril smiles”, and \( r \) is the proposition “Avril is happy”.
  - Then \( p \rightarrow q \) is the proposition “Avril smiles when she does logic”; \( q \rightarrow r \) is the proposition “Avril is happy when she smiles”; \( p \rightarrow r \) is the proposition “Avril is happy when doing logic”.

The Disjunctive Syllogism

- **Definition**
  - The *disjunctive syllogism* is the rule of inference based on the tautology \( (p \lor q) \land \neg p \rightarrow q \).

- **Example**
  - “If this lecture finishes on time or I’m wearing red socks (or both), and this lecture doesn’t finish on time, then I’m wearing red socks.”

Resolution

- **Definition**
  - *Resolution* is the rule of inference based on the tautology \( (p \lor q) \land (\neg p \lor r) \rightarrow q \lor r \).

- **Example**
  - “Jasmine is skiing, or it isn’t snowing. Also, it is snowing, or Bart is playing hockey. Therefore Jasmine is skiing or Bart is playing hockey.”
Valid Arguments

• An argument is *valid*, if whenever the hypotheses are true, the conclusion is true.
  – it is irrelevant whether the hypotheses are actually true.

• Example
  Let \( x \in \mathbb{R} \).
  If \( x^2 \geq 0 \), then \( x \geq 0 \). \( x < 0 \), so \( x^2 < 0 \).
  This *is* a valid argument (modus tollens). However the initial hypothesis “if \( x^2 \geq 0 \), then \( x \geq 0 \)” is untrue.

• To demonstrate that a complex argument is valid, we should break down the argument into steps.
  – each step should correspond to one rule of inference.

Example Proof

• Example
  – Construct a valid arguments that leads from the hypotheses below to the conclusion “we will be home by sunset.”
    • Hypothesis 1: “It is not sunny this afternoon and it is colder than yesterday”.
    • Hypothesis 2: “We will go swimming only if it is sunny”.
    • Hypothesis 3: “If we do not go swimming, then we will take a canoe trip”.
    • Hypothesis 4: “If we take a canoe trip, then we will be home by sunset”.
  – We will start by labelling our propositions.
    • Let \( p \) be the proposition “it is sunny this afternoon”
    • Let \( q \) be the proposition “it is colder than yesterday”.
    • Let \( r \) be the proposition “we will go swimming”.
    • Let \( s \) be the proposition “we will take a canoe trip”.
    • Let \( t \) be the proposition “we will be home by sunset”.
  – We may now express our hypotheses as:
    1. \( \neg p \land q \) (Hypothesis 1)
    2. \( r \rightarrow p \)
    3. \( \neg r \rightarrow s \)
    4. \( s \rightarrow t \)
  – Our proof can now be broken down into the following steps
    1. \( \neg p \land q \) (Hypothesis 1)
    2. \( \neg p \) (Simplification)
    3. \( \neg r \) (Modus tollens using Hypothesis 2)
    4. \( s \) (Modus ponens using Hypothesis 3)
    5. \( t \) (Modus ponens using Hypothesis 4)
  – In English, these steps are:
    • “It is not sunny this afternoon and it is colder than yesterday. Therefore it is not sunny this afternoon.”
    • “It is not sunny this afternoon, and we will go swimming only if it is sunny. Therefore we will not go swimming.”
    • “We will not go swimming, and if we do not go swimming we will take a canoe trip. Therefore we will take a canoe trip.”
    • “We will take a canoe trip, and if we take a canoe trip we will be home by sunset. Therefore we will be home by sunset.”
Inference for Quantified Statements

• So far, our rules of inference have worked with propositional logic. Now for some predicate logic.
• There are two main extra rules of inference here:
  – you may wish to go from the general to the particular, e.g. “all people have a heart” to “George has a heart”.
  – you may wish to go from the particular to the general, e.g. “George has a heart” to “somebody has a heart”.
• Definition
  – Universal instantiation is the rule of inference based on the tautology \( \forall x P(x) \rightarrow P(c) \)
• Example
  – Let the universe of discourse be the set of men. Let \( P(x) \) be the predicate “\( x \) is mortal”.

Example Proof

• Example
  – Construct a valid argument that leads to the conclusion “George is crazy” from the hypotheses below. Use the set of people as the universe of discourse.
    • Hypothesis 1: “George is a lecturer”
    • Hypothesis 2: “All lecturers are crazy”
  – First, we will express the hypotheses in symbolic form.
    • Let \( P(x) \) be the predicate “\( x \) is a lecturer”.
    • Let \( Q(x) \) be the predicate “\( x \) is crazy”.
    • Hypothesis 1: \( P(\text{George}) \).
    • Hypothesis 2: \( \forall x ( P(x) \rightarrow Q(x) ) \).

Example Proof

1. \( \forall x ( P(x) \rightarrow Q(x) ) \) (Hypothesis 2)
2. \( P(\text{George}) \rightarrow Q(\text{George}) \) (Universal instantiation)
3. \( Q(\text{George}) \) (Modus ponens, using Hypothesis 1)

– In English, these steps are:
  • “All lecturers are crazy. Therefore if George is a lecturer, he is crazy”.
  • “George is a lecturer, and if George is a lecturer, he is crazy. Therefore George is crazy.”
Some Common Errors

- Errors in logical reasoning are called **fallacies**.
- Two fallacies are very common
  - the fallacy of affirming the conclusion.
  - the fallacy of denying the hypothesis.
- These occur because we use a "rule" that is not based on a tautology, and therefore not actually a rule of inference.
- Affirming the conclusion
  \[(p \rightarrow q) \land q \rightarrow p\] is not a tautology
  - check when \(p\) false and \(q\) true.
- **Example**
  - "If I study very hard, I will get an A. I got an A. Therefore I studied very hard." [a fallacy!]
- Denying the hypothesis
  \[(p \rightarrow q) \land \neg p \rightarrow \neg q\] is not a tautology
  - check when \(p\) false and \(q\) true.
- **Example**
  - "If I study very hard, I will get an A. I didn’t study very hard. Therefore I won’t get an A." [a fallacy!]
- Another common error: what is wrong with this argument?
  1. \(a = b\)
  2. \(2a = a + b\)
  3. \(2a - 2b = a - b\)
  4. \(2(a - b) = 1(a - b)\)
  5. \(2 = 1\)
- This fallacy will become clear if we make our hypotheses explicit.
  - Let the universe of discourse be the set of real numbers.
  - The final step is based on the "cancellation" axiom:
    \[\forall x \forall y \forall z \ ( (xy = xz) \land x \neq 0 \rightarrow (y = z) ) .\]
  - The fallacy lies in the last step of the argument.
    - It is not a valid application of the cancellation axiom, since \(a - b = 0\).
  - The danger here is that we are relying on our faulty recollection of a tautology
    - we remembered \(\forall x \forall y \forall z \ ( (xa = xb) \land x \neq 0 \rightarrow (a = b) )\) as \(\forall x \forall y \forall z \ ( (xa = xb) \rightarrow (a = b) )\), which is not a tautology.
      - check with \(x = 0\), \(a = 1\), \(b = 2\).
- What’s the rule of inference used here?
  (a) Kangaroos live in Australia and are marsupials. Therefore kangaroos are marsupials.
  (b) If I work all night, I will answer all the questions. If I answer all the questions, I will know the subject. Therefore, if I work all night, I will know the subject.
- Valid or invalid arguments? For valid arguments, what rule(s) of inference have been used?
  (a) If \(x^2\) is irrational, then \(x\) is irrational. Therefore if \(x\) is irrational, it follows that \(x^2\) is irrational.
  (b) If \(x^2\) is irrational, then \(x\) is irrational. The number \(\pi^2\) is irrational. Therefore the number \(\pi\) is irrational.
Summary

• This lecture has covered
  – What is a theorem, lemma, and proof?
  – Valid arguments.
  – Inference for quantified statements.
  – Common errors.
• This was the last lecture on logic.
• In the next lecture we will look at the tools necessary to analyse algorithms.