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The Complexity of Multiple Wordlength Assignment

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Abstract—This note discusses the multiple wordlength assignment problem for the design of custom digital signal processing (DSP) parallel processors. It is demonstrated that this assignment problem is NP-hard.

Keywords—Digital signal processing, Multiple wordlength, Precision, Computational complexity.

1. INTRODUCTION

A first stage in algorithm design for digital signal processing (DSP) systems is to select the system wordlength (precision) necessary to implement the algorithm while maintaining acceptable signal distortion due to roundoff or truncation noise. In the past, it has been common practice to select a single such wordlength, which is used to represent all signals as the algorithm executes. This design approach results from implementation in a single processor of fixed wordlength. However, the move away from a single arithmetic computational unit towards forms of parallel processing has shown that the required precision of a calculation is an important dimension in designing fast, low-power, and area-efficient processors [1]. In custom hardware implementations, such as those on field programmable gate arrays (FPGAs) this is a particularly important trend, since the hardware designer has the freedom to construct datapaths with precision customized for each algorithm [2].

It is well known that the variance of the roundoff error injected into a calculation due to roundoff from wordlength x_1 to x_2 can be approximated by $(1/12)2^{-2x_2}$, for $x_1 \gg x_2$ [3], and that this error variance can then be scaled through L_2 scaling [4] to estimate the error variance

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at the observable outputs of a linear time-invariant (LTI) system. For a large enough dynamic range of the rounded signal values, it can be assumed that the errors injected into a system at different roundoff points are uncorrelated. Let there be a total of m observable system outputs, and n points of truncation or roundoff within the system. Further, let the wordlength at point i with $1 \leq i \leq m$ be x_i. Then the observable error variance at output j with $1 \leq j \leq m$ of the system can be estimated using (1). Here b_{ij} is a coefficient derived from an analysis of the transfer function of the system in question [4], and $b_{ij} \geq 0$ for all i,j with $1 \leq i \leq n$ and $1 \leq j \leq m$.

$$\sigma_j^2 = \sum_{i=1}^{n} b_{ij} 2^{-2x_i} \tag{1}$$

The silicon area consumed by a given DSP implementation can be estimated by the area of each operation, as control overheads tend to be limited for this type of design [5]. The area of the circuit in question can therefore be estimated by (2), which is justified on the grounds that the area consumed by linear system implementation tends to be dominated by constant-coefficient multipliers and registers, whose area scales roughly proportionally with their input wordlength. Note also that these scaling factors satisfy $a_i \geq 0$ for all i with $1 \leq i \leq n$.

$$\text{Area} = \sum_{i=1}^{n} a_i x_i. \tag{2}$$

The traditional problem of choosing a single uniform wordlength to satisfy constraints on area and error power is therefore relatively trivial: one must choose the wordlength satisfying the most constraining inequality in (1), which will in turn minimize the circuit area.

With the rise of multiple precision algorithms, and the multiple precision design approach [2], the choice of a set x_i of word lengths becomes a significantly harder problem. In the remainder of this note, we formulate this wordlength selection as a decision problem: *"can a set of wordlengths be found to satisfy both the conflicting requirements of adequately low signal distortion observable at each system output, and adequately low area consumption?"* For each output j with $1 \leq j \leq m$, the maximum allowable output error variance is denoted by B_j. The maximum allowable area is denoted by A.

PROBLEM. MULTIPLE WORDLENGTH SELECTION.

INSTANCE. Nonnegative real numbers a_i with $1 \leq i \leq n$; nonnegative real numbers b_{ij} with $1 \leq i \leq n$ and $1 \leq j \leq m$; a nonnegative real number A; nonnegative real numbers B_j with $1 \leq j \leq m$.

QUESTION. Do there exist n positive integers x_i with $1 \leq i \leq n$ such that $\sum_{i=1}^{n} a_i x_i \leq A$ and such that $\sum_{i=1}^{n} b_{ij} 2^{-2x_i} \leq B_j$ for all j with $1 \leq j \leq m$?

The contribution of this short note is to establish the NP-hardness of the *multiple wordlength selection* problem. Hence, the *multiple wordlength selection* problem is computationally intractable and does not possess a fast solution algorithm unless the complexity classes P and NP coincide.

2. THE NP-HARDNESS PROOF

NP-hardness of the *multiple wordlength selection* problem is established by a polynomial time reduction from the *three-satisfiability* problem. The *three-satisfiability* problem is a well-known and fundamental NP-hard problem [6], and it is defined as follows.

PROBLEM. THREE SATISFIABILITY.

INSTANCE. A set $U = \{u_1, \ldots, u_\ell\}$ of logical variables; a set $C = \{c_1, \ldots, c_k\}$ of three-literal clauses over U. (A *literal* is a negated or an unnegated logical variable; a *three-literal clause* is the disjunction of three literals.)

QUESTION. Does there exist a truth setting for U that satisfies all clauses in C?

We will now show how to translate any instance (U, C) of *three-satisfiability* within polynomial time into an instance I of *multiple wordlength selection* such that instance (U, C) has answer YES if and only if instance I has answer YES. Clearly, this will establish the NP-hardness of *multiple wordlength selection*. In our argument, we will mainly be working with the wordlengths one and two. These wordlengths are so small that the errors injected can lose their flat (white) spectrum, which is assumed for the scaling. However, our argument can easily be modified to yield NP-hardness of the problem variant with larger wordlengths, i.e., the case where the feasible wordlengths are bounded from below by some fixed constant ω.

Now consider an instance (U, C) of *three-satisfiability*. For every logical variable u_i in U with $1 \le i \le \ell$, we create two corresponding wordlengths x_{2i} and x_{2i-1} in instance I. Intuitively speaking, the wordlength x_{2i} corresponds to the unnegated literal u_i, and the wordlength x_{2i-1} corresponds to the negated literal $\overline{u_i}$. Moreover, the situation $x_{2i} = 2$ and $x_{2i-1} = 1$ will correspond to $u_i =$ TRUE, and $x_{2i} = 1$ and $x_{2i-1} = 2$ will correspond to $u_i =$ FALSE.

- We set $a_i = 1$ for $1 \le i \le 2\ell$, and we set $A = 3\ell$.
- For every logical variable u_i with $1 \le i \le \ell$, we introduce a corresponding inequalities of Type I (this is done by setting the coefficients b_{ij} and B_j to appropriate values): $2^{-2x_{2i}} + 2^{-2x_{2i-1}} \le 5/16$.
- For every clause c in C, we introduce a corresponding inequality of Type II. If the logical variable u_i occurs as an unnegated literal in c, then the left-hand side of the inequality contains the term $2^{-2x_{2i}}$ with coefficient 1. And if the logical variable u_i occurs as a negated literal in c, then the left-hand side of the inequality contains the term $2^{-2x_{2i-1}}$ with coefficient 1. All other coefficients are set to 0. The right-hand side of each such inequality of Type II equals 9/16.

  For instance, according to these rules, the clause $c = (u_1 \vee \overline{u_2} \vee u_3)$ translates into the inequality $2^{-2x_2} + 2^{-2x_3} + 2^{-2x_6} \le 9/16$.

This completes the description of the instance I of the *multiple wordlength selection* problem. Note that $n = 2|U| = 2\ell$ and that $m = |U| + |C| = \ell + k$. Clearly, instance I can be computed in polynomial time.

LEMMA 1. *If the three-satisfiability instance (U, C) is satisfiable, then the instance I of multiple wordlength selection has a solution.*

PROOF. Consider a satisfying truth setting for (U, C). If $u_i =$ TRUE, then we set $x_{2i} = 2$ and $x_{2i-1} = 1$. If $u_i =$ FALSE, then we set $x_{2i} = 1$ and $x_{2i-1} = 2$. Then, for every u_i, the corresponding inequality $2^{-2x_{2i}} + 2^{-2x_{2i-1}} \le 5/16$ of Type I is satisfied even with equality. Now consider an inequality of Type II. The corresponding clause c contains at least one TRUE literal, and the term that represents this TRUE literal in the inequality has value $2^{-4} = 1/16$. The other two terms have values 1/16 (if the corresponding literal is TRUE) or 1/4 (if the corresponding literal is FALSE). Since these three terms add up to at most $1/16 + 1/4 + 1/4 = 9/16$, also all inequalities of Type II are satisfied. ∎

LEMMA 2. *If the instance I of multiple wordlength selection has a solution, then the three-satisfiability instance (U, C) is satisfiable.*

PROOF. Since all wordlengths x_{2i-1} and x_{2i} are positive integers, $x_{2i-1} \ge 1$ and $x_{2i} \ge 1$. From the inequality $2^{-2x_{2i}} + 2^{-2x_{2i-1}} \le 5/16$ of Type I, we get that at most one of x_{2i-1} and x_{2i} may be equal to 1, and the other one must be at least 2. Therefore, $x_{2i-1} + x_{2i} \ge 3$ and $\sum_{i=1}^{2\ell} x_i \ge 3\ell$. But the area bound in the *multiple wordlength selection* instance states that $\sum_{i=1}^{2\ell} x_i \le 3\ell$. Putting things together, $\sum_{i=1}^{2\ell} x_i = 3\ell$, and for every i exactly one of x_{2i-1} and x_{2i} equals 1 while the other one equals 2.

Consider the following truth setting for U: if $x_{2i} = 2$ and $x_{2i-1} = 1$ then we set $u_i =$ TRUE. And if $x_{2i} = 1$ and $x_{2i-1} = 2$ then we set $u_i =$ FALSE. Suppose that this truth setting yields an unsatisfied clause c: Then in this clause c all literals are FALSE, and in the corresponding inequality of Type II the left-hand side equals $2^{-2} + 2^{-2} + 2^{-2} = 3/4 > 9/16$. But then this inequality is violated. ∎

THEOREM 3. *The multiple wordlength selection problem is NP-hard.*

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