Rational Transfer functions and the Bode Plot

The transfer function

- The transfer function is the Fourier transform of the impulse response
- Filters we can make have a rational transfer function: the transfer function is is a ratio of two polynomials with real coefficients.

  (strictly speaking this is called the “Padé approximation”: it states that any real function can be approximated by a rational function. The higher the degree of the polynomials the closer the approximation can be made)

  The notation is s=jω. The signals assumed to be sinusoid: $V = V_0 e^{j\omega t+\phi}$

$$H(s) = \frac{P_n(s)}{Q_n(s)} = \frac{\sum_{k=0}^{n} a_k s^k}{\sum_{k=0}^{m} b_k s^k} = \frac{a_n(s-z_1)(s-z_2)\cdots(s-z_n)}{b_m(s-p_1)(s-p_2)\cdots(s-p_m)}$$

- The roots z_k of the numerator polynomial are called the “zeroes” of H
- The roots p_k of the denominator polynomial are called the “poles” of H
- The pole positions on the complex frequency plane entirely determine the filter properties.
- Note that since s=jω the denominator is seldom zero unless it has pure imaginary roots

The Bode plot – some maths

Complex numbers can be written in rectangular or polar representation:

$$z = x + iy = \mathrm{Re}^{j\theta}$$

The natural logarithm function works correctly with complex argument:

$$\ln z = \ln\left(\mathrm{Re}^{j(\theta+2n\pi)}\right) = \ln R + j\theta = \ln\left|z\right| + j\arg z + j2\pi n$$

It follows that, if we restrict the phase to a circle,

$$\mathrm{Im}\ln z = \arg z$$

Since H(s) is rational, we can write:

$$\ln H(s) = \ln\left(\frac{a_n \prod_{k=1}^{n}(s - z_k)}{b_m \prod_{k=1}^{m}(s - p_k)}\right) = \ln\frac{a_n}{b_m} + \sum_{k=1}^{n}\ln(s - z_k) - \sum_{k=1}^{m}\ln(s - p_k)$$

The Bode plot – Magnitude plot

Since H(s) is rational, we can write:

$$\ln\left|H(s)\right| = \ln\left|\frac{a_n}{b_m}\right| + \left(\sum_{k=1}^{n}\ln\left|s-z_k\right|\right) - \left(\sum_{k=1}^{m}\ln\left|s-p_k\right|\right)$$

The log of the transfer function is the sum of terms of the form $\ln\left|s-x\right|$
Each term takes two simple limits for high and low frequencies respectively:

$$\lim_{|s|>>x}\ln\left|s-x\right| = \ln\left|s\right| = \ln\omega \quad , \quad \lim_{|s|<<x}\ln\left|s-x\right| = \ln\left|x\right|$$

Each of the terms in the numerator contributes a constant at low frequencies and a line of slope 1 at high frequencies. The corner is at w=z_k

Each of the terms in the denominator contributes a constant at low frequencies and a line of slope -1 at high frequencies. The corner is at w=p_k

The Bode plot – Phase plot

Phase plot: Make a a linear – log plot of the phase (i.e. the imaginary part of the logarithm of the transfer function) versus the log of frequency.

$$\arg H(s) = \operatorname{Im}\ln H(s) = \arg\frac{a_n}{b_m} + \sum_{k=1}^{n}\arg(s - z_k) - \sum_{k=1}^{m}\arg(s - p_k)$$

Each term in the numerator contributes (remember s is imaginary!)

$$\lim_{\omega\to 0}\arg(s - z_n) = 0$$

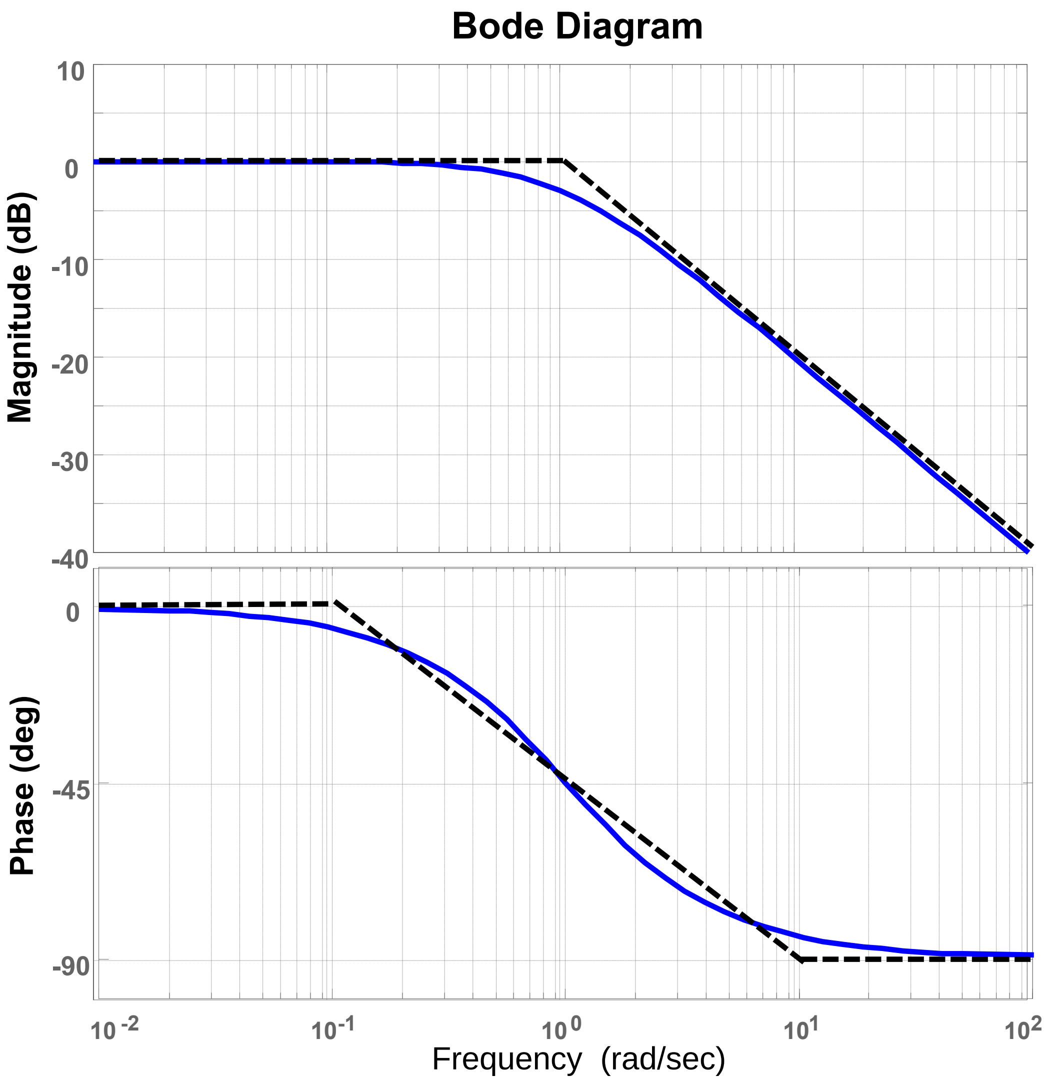
$$\lim_{\omega\to\infty}\arg(s - z_n) = \pi/2$$

$$\lim_{\omega\to z_n}\arg(s - z_n) = \tan^{-1}\frac{\omega}{z_n}\bigg|_{\omega\simeq z_n} \simeq \frac{\pi}{4} + \frac{1}{2}\left(\frac{\omega}{z_n} - 1\right) \simeq \frac{\pi}{4} + \frac{1}{2}\ln\frac{\omega}{z_n}$$

since $\ln(1+\varepsilon) \simeq \varepsilon$. in this case, $\varepsilon = \frac{\omega}{z_n} - 1$. also note that $\log x = \ln x / \ln 10$

- **each term has linear asymptote at 0 radians for small frequencies**
- **each term has linear asymptote at π/2 radians at high frequencies**
- **Each term contributes a linear (-/+) slope near the pole or zero**
- **a factor of 10 in frequency adds/subtracts 45 degrees to phase**

Bode plot example -poles



Magnitude plot:

Break point at pole

Slope: -20dB per decade increase in frequency

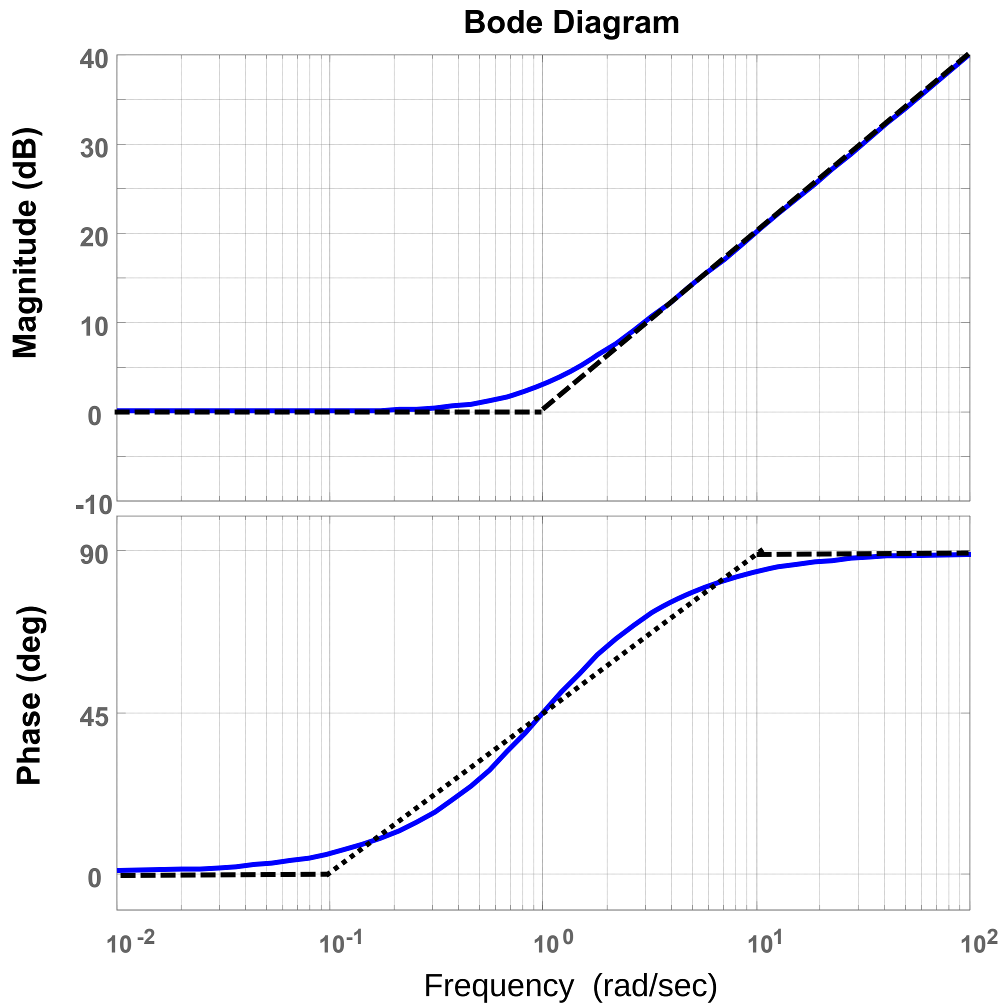
Phase plot:

-45 degrees at pole

Slope:
-45 degrees per factor of 5 increase in frequency

Often approximated to -90 degrees/ 2 decades

Bode plots - Zeroes



Magnitude plot:

Break point at pole

Slope: 20dB per decade increase in frequency

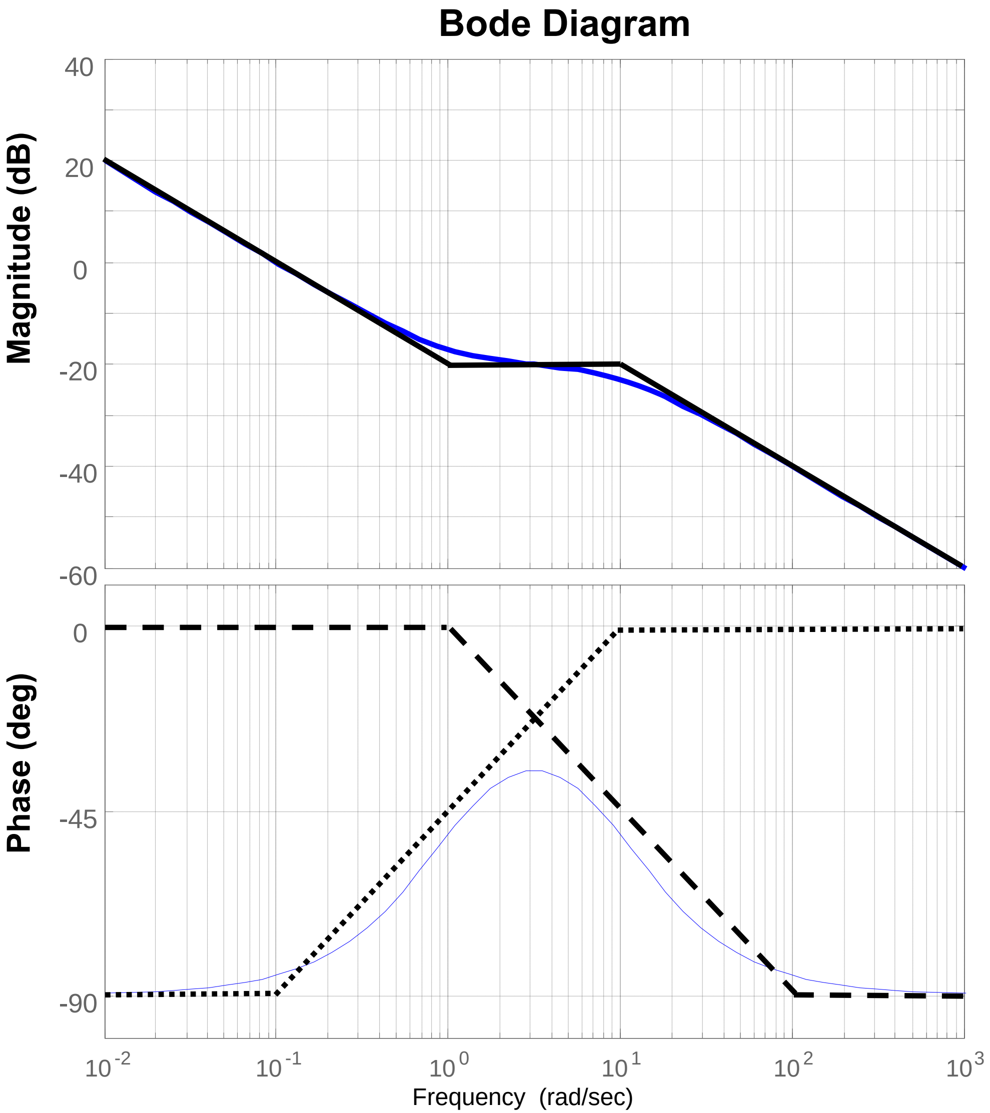
Phase plot:

45 degrees at pole

Slope:
45 degrees per factor of 5 increase in frequency

Often approximated to 90 degrees/ 2 decades

Bode plot example: 2 poles 1 zero



$$H(f) = \frac{(s+1)}{s(s+10)}$$