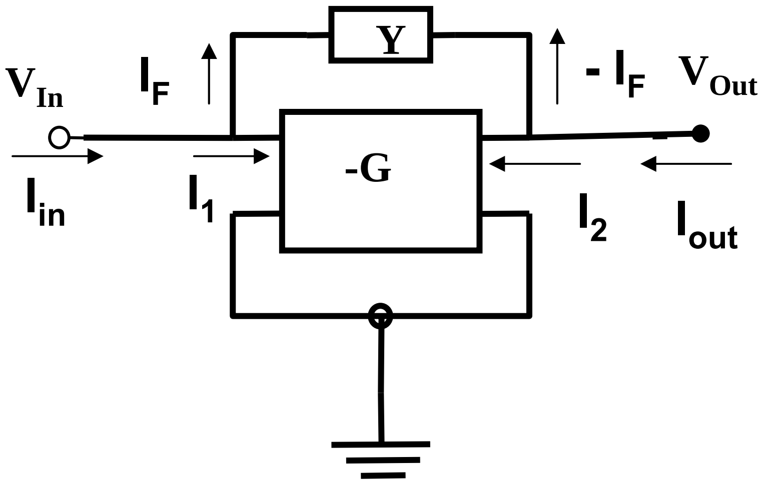
The Miller Theorem

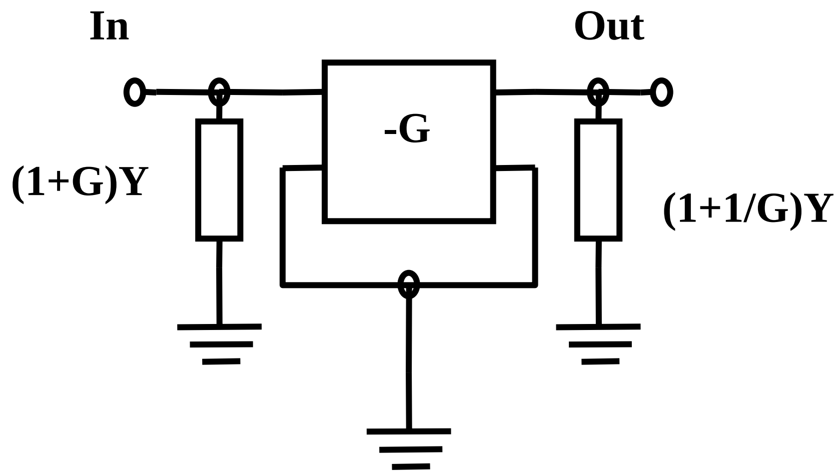
and the

Frequency Response of the Common Emitter Amplifier

A useful approximation: The Miller Theorem

- Consider a shunt admittance connected between the input and output of an inverting voltage amplifier of gain G.





- Assuming that Y does not affect the gain, KCL at the input says that:

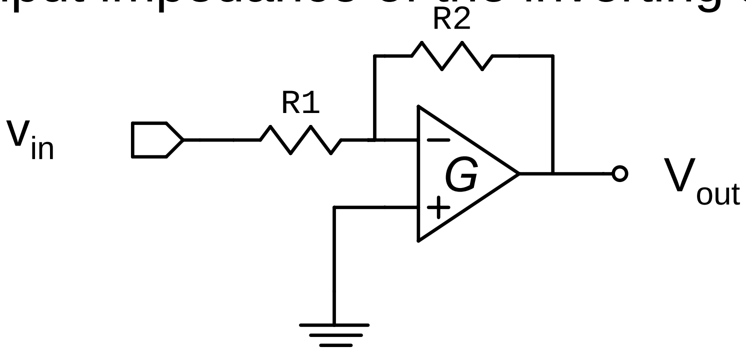
$$i_{in} = i_1 + i_F = i_1 + (v_{in} - v_{out})Y = i_1 + \left[(1+G)Y\right]v_{in}$$

$$i_{out} = i_2 - i_F = i_2 - (v_{in} - v_{out})Y = i_2 + \left[(1+1/G)Y\right]v_{out}$$

- This is the same current as we would get if we connected Y(1+G) in parallel to the input and Y(1+1/G) in parallel to the output (on the right)
- **Note that this is an approximation; Y always changes the gain, unless $Z_{out} // Z_L = 0$. Nonetheless, very often it is a good approximation.**

An application of the Miller theorem

Input impedance of the inverting amplifier (if op-amp has zero Y_{in}):



$$Z_{in} = R_1 + \frac{R_2}{(1+G)}$$

We can even use the Miller theorem to calculate the gain of this circuit for finite op-amp gain G:

$$A_v = \frac{v_{out}}{v_{in}} = -G\frac{R_2/(G+1)}{R_1 + R_2/(G+1)} = \frac{-GR_2}{R_1G + R_1 + R_2}\ , \quad \lim_{G\to\infty} A_v = \frac{-R_2}{R_1}$$

This is the correct answer for the closed loop gain as we shall later see!

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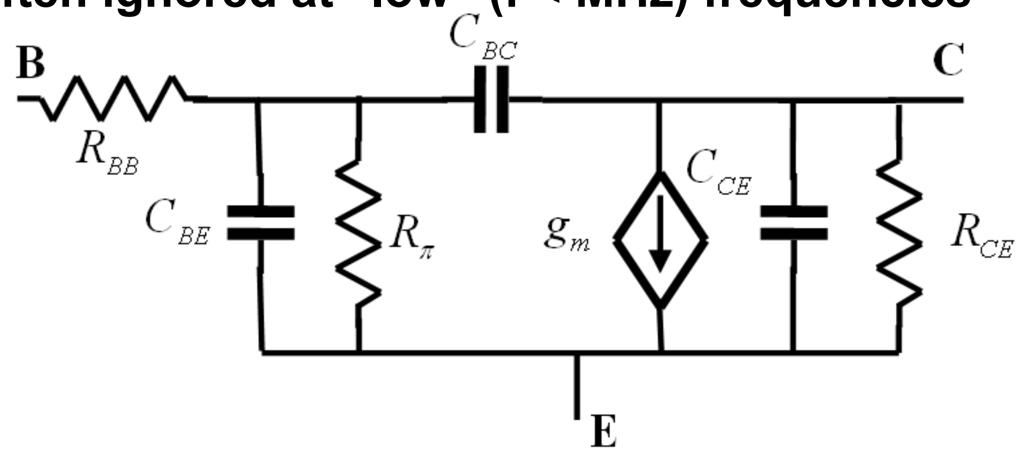
The small signal model the of BJT

- The model below is useful to frequencies of up to 100s of MHz
- The model is strictly valid for B-E voltage variations smaller than V_T=25mV.
- To analyse a circuit:
  - *Each transistor in a circuit is replaced by this model*
  - *The "base spreading resistance" R_{BB} is lumped with the thevenin impedance of the signal source. It is not shown explicitly in these notes.*

What does R_{BB} do:

- R_{BB} is responsible for most of the noise of transistor amplifiers
- R_{BB} is responsible for the power gain rolling-off at high frequencies
- R_{BB} is often ignored at "low" (f < MHz) frequencies



BJT model: impedances and transconductance

- **The large signal BJT equation gives the transconductance:**

$$I_C = I_0\left(e^{V_{be}/V_{th}} - 1\right)\left(1 + \frac{V_{CE}}{V_A}\right) \simeq I_0 e^{V_{be}/V_{th}}$$

$$V_{th} = \frac{k_B T}{e} \simeq 26\ \text{mV at room temperature}$$

$$g_m = \frac{\partial I_C}{\partial V_{be}} \simeq \frac{I_C}{V_{th}}$$

- **However, since $I_E=(1+1/\beta)I_C$,**

$$R_\pi = \frac{\partial V_{BE}}{\partial I_B} = \beta V_T / I_C = \beta / g_m$$

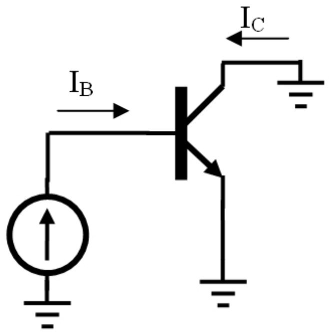
- **Same equation gives output conductance:**

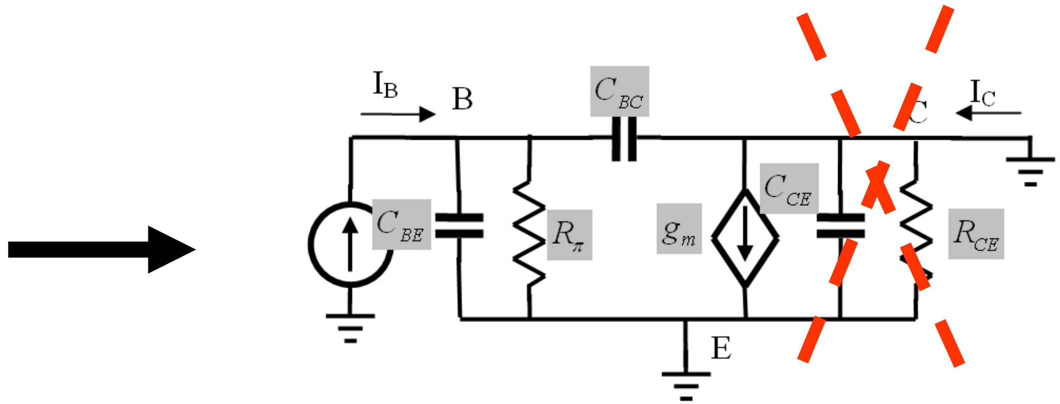
$$G_{CE} = 1/R_{CE} = \frac{\partial I_C}{\partial V_{CE}} = \frac{I_C}{V_A}$$

BJT model Capacitances

• **BC junction is reverse biased:** – ***depletion capacitance*** – ***overlap capacitance***	$C_{BC} = C_{BC0} + C_j$
• **BE junction is forward biased:** – ***depletion capacitance*** – ***storage (diffusion capacitance)*** – ***overlap capacitance***	$C_{BE} = C_{BE0} + C_j + C_D$
• **CE capacitance is small** – ***overlap capacitance***	$C_{CE} \simeq C_{CE0}$
• **Fringing/Overlap capacitances:**	$C_{BC0}\ ,\ C_{BE0}\ ,\ C_{CE0}$
• **Junction capacitance:** – ***Reverse biased diode*** – ***M=1/2-1/3, from doping*** – ***Φ approx 0.8V*** – ***M, Φ usually determined from two bias points***	$C_j = \dfrac{C_0}{\left(1 - \dfrac{V_{BE}}{\Phi}\right)^M}$
• **Diffusion capacitance** – ***from transit time***	$C_D = g_m \tau_T \quad , \quad \tau_T = 1/2\pi f_T$

An important figure of merit: the BJT Current gain and the transit frequency f_T





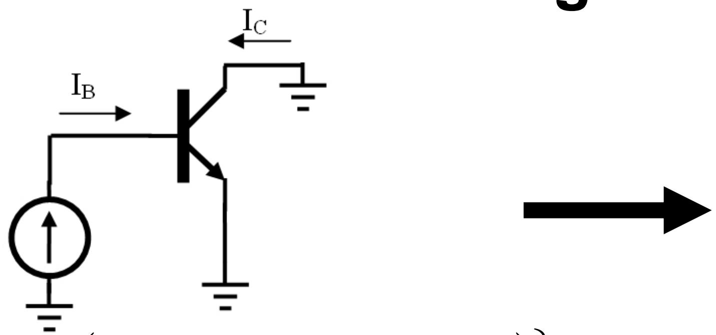
The transistor is connected as a current amplifier:
- Driven by a current source
- Driving a current meter

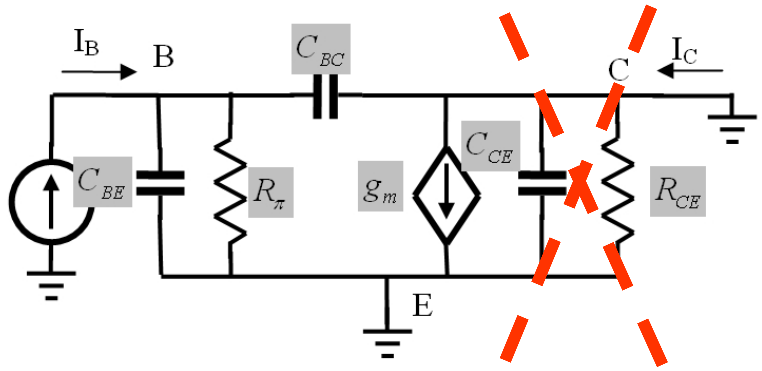
f_T is the frequency at which the current gain drops to unity.

f_T is usually specified by the transistor manufacturer as a figure of merit.

f_T is a rough estimate of the highest frequency at which the transistor can be used as an amplifier. Typically fT / 10 is this highest frequency.

An important figure of merit: the BJT Current gain and the transit frequency f_T





$$\left.\begin{aligned} i_B &= v_{BE}\left(s\left(C_{BE}+C_{BC}\right)+1/R_\pi\right) \\ i_C &= g_m v_{BE} \end{aligned}\right\} \Rightarrow$$

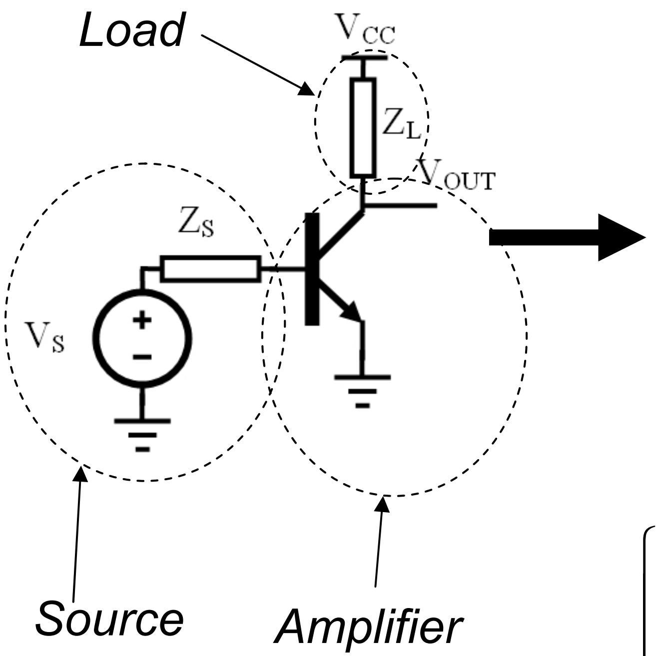
$$\frac{i_C}{i_B} = "h_{fe}" = \frac{g_m}{s\left(C_{BE}+C_{BC}\right)+1/R_\pi} = \frac{g_m R_\pi}{1+sR_\pi\left(C_{BE}+C_{BC}\right)} = \frac{\beta_0}{1+s\beta_0/2\pi f_T}$$

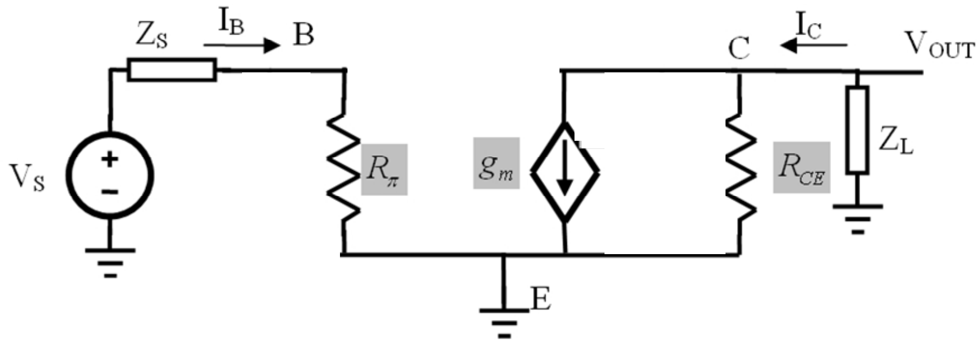
where $f_T = \dfrac{\beta_0}{2\pi R_\pi\left(C_{BE}+C_{BC}\right)} = \dfrac{g_m}{2\pi\left(C_{BE}+C_{BC}\right)} \simeq \dfrac{g_m}{2\pi C_{BE}}$, since: $R_\pi = \dfrac{\beta_0}{g_m}$

the current gain is $h_{fe}(f) = \dfrac{\beta_0}{1+j\beta_0 f/f_T} \Rightarrow$ $\lim\limits_{f>f_T/\beta_0}\left|h_{fe}\right| = \dfrac{f_T}{f}$ with $f_T = \dfrac{1}{2\pi\tau_T}$

f_T allows the easy calculation of C_{BE} + C_{BC} since g_m only depends on the collector bias current! Note that C_{BE}>>C_{BC} so knowing f_T defines C_{BE}

The Common Emitter amplifier



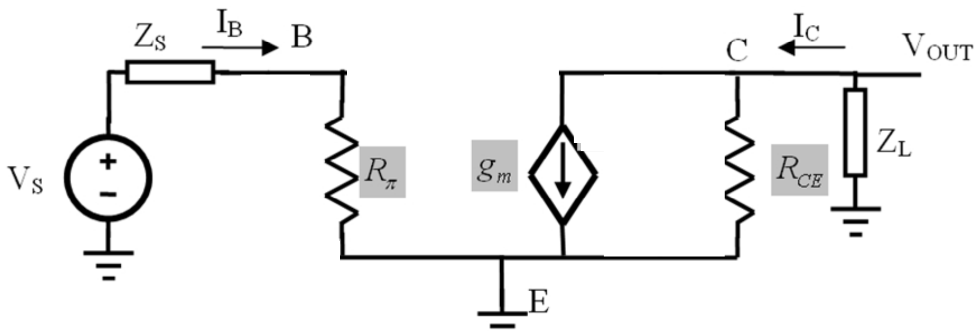


Model parameters:

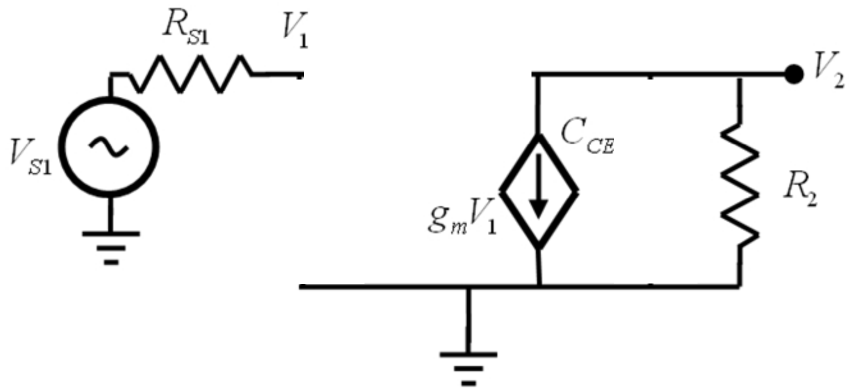
$$\begin{cases} g_m = \dfrac{\partial I_C}{\partial V_{BE}} = \dfrac{I_C}{V_T} \\ R_\pi = \left(\dfrac{\partial I_B}{\partial V_{BE}}\right)^{-1} = \left(\dfrac{\partial\left(I_C / \beta\right)}{\partial V_{BE}}\right)^{-1} = \beta\left(\dfrac{\partial I_C}{\partial V_{BE}}\right)^{-1} = \dfrac{\beta}{g_m} \\ R_{CE} = \left(\dfrac{\partial I_C}{\partial V_{CE}}\right)^{-1} \simeq \dfrac{V_A}{I_C} \end{cases}$$

$V_T = kT / q =$ 25mV at 290K=17C

The Common Emitter amplifier (2)



Combine V_S, Z_S, R_π *into a Thevenin circuit:*



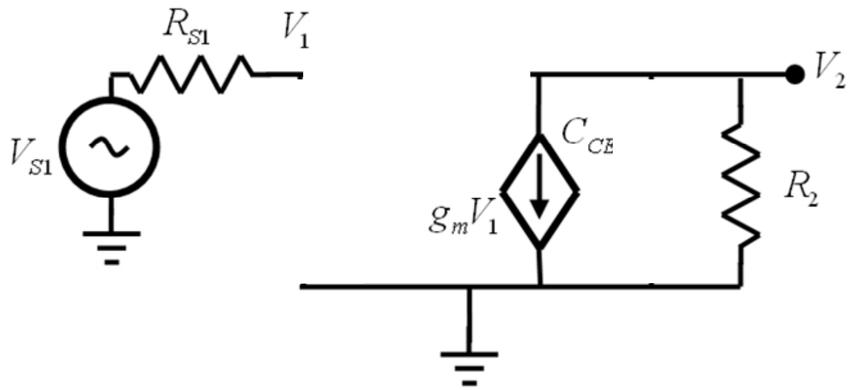
$$V_{S1} = V_S \frac{R_\pi}{R_\pi + Z_S} = \frac{V_S}{1 + Z_S Y_{in}}$$

$$R_{S1} = R_\pi // Z_S = \frac{R_\pi Z_S}{R_\pi + Z_S} = \frac{Z_S}{1 + Z_S Y_{in}}$$

Consider R_{CE} and Z_L together: $R_2 = R_{CE} // Z_L = \dfrac{R_{CE} Z_L}{R_{CE} + Z_L} = \dfrac{Z_L}{1 + Z_L Y_{out}}$

and note that $Z_S = R_S + R_{BB}$

The Common Emitter amplifier (3)



Gain:

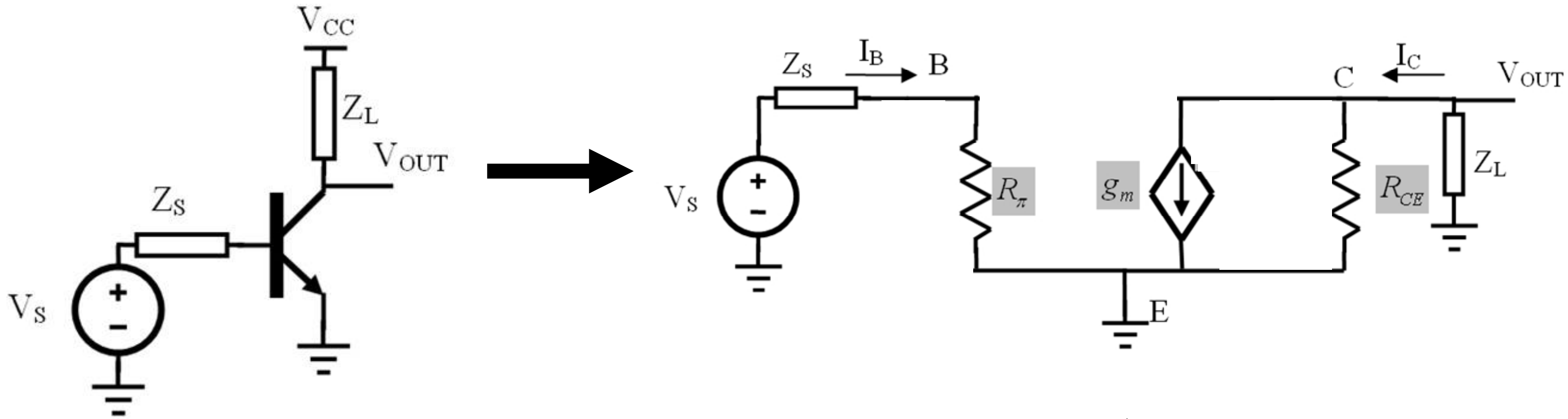
$$A_v = \frac{v_{out}}{v_s} \equiv \frac{\partial V_2}{\partial V_s} = -\frac{V_{s1}}{V_s} g_m R_2 = \frac{1}{\left(1+R_s Y_{in}\right)} \frac{-g_m Z_L}{\left(1+Z_L Y_{out}\right)}$$

If Y_{in}= 0 and Y_{out} = 0 this is the familiar:

$$A_v = \frac{v_{out}}{v_s} \equiv -g_m Z_L$$

Note that lowercase letters represent small signal variations, i.e. differentials

The Common Emitter amplifier (4): Input and output impedance



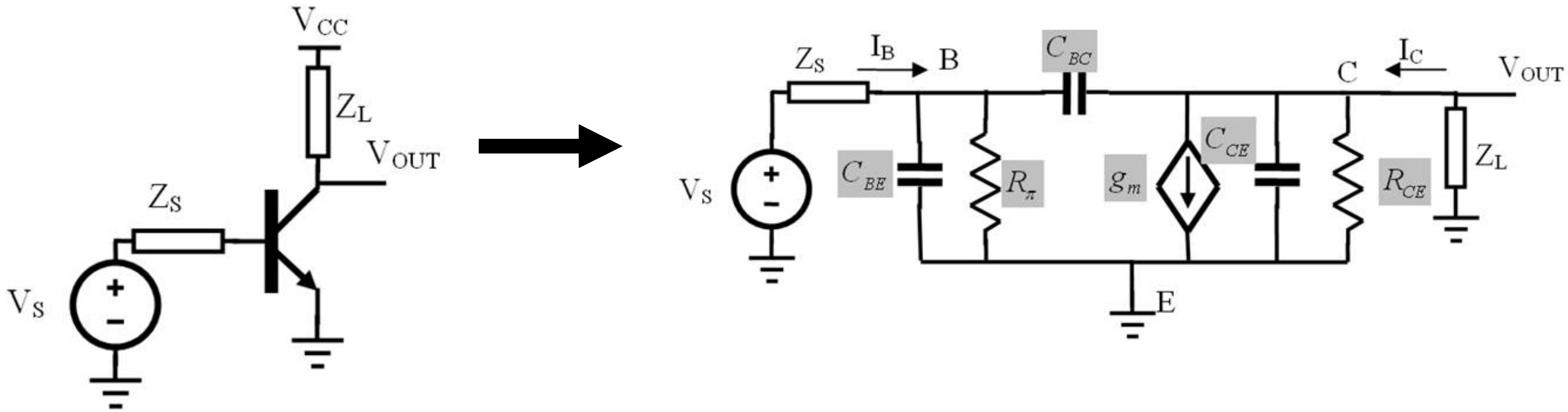
Input Impedance:

$$Z_{in} = \frac{\partial V_{BE}}{\partial I_B} = \left(\frac{\partial I_B}{\partial V_{BE}}\right)^{-1} = R_\pi$$

Output Impedance:

$$Z_{out} = \frac{\partial V_C}{\partial I_{out}} = \left(\frac{\partial I_C}{\partial V_{CE}}\right)^{-1} = R_{CE}$$

Frequency response of the Common Emitter amplifier



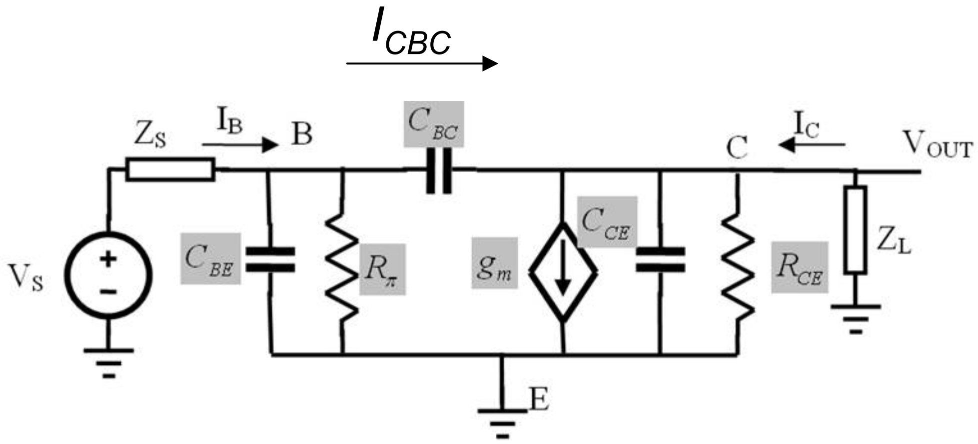
If we momentarily neglect C_{BC} this would be the same as before but with:

Input Impedance:
$$Z_{in} = \frac{1}{Y_{in}} = R_\pi // C_{BE} = \frac{R_\pi}{1 + j\omega C_{BE} R_\pi}$$

Output Impedance:
$$Z_{out} = \frac{1}{Y_{out}} = R_{CE} // C_{CE} = \frac{R_{CE}}{1 + j\omega C_{CE} R_{CE}}$$

We still have:
$$A_V = \frac{v_{out}}{v_s} = \frac{1}{(1 + R_s Y_{in})} \frac{-g_m Z_L}{(1 + Z_L Y_{out})}$$ *and* $$\frac{v_{out}}{v_B} = G = \frac{-g_m Z_L}{(1 + Z_L Y_{out})}$$

Dealing with C_{BC}



In the absence of C_{BC} we know that:

$$\frac{v_{out}}{v_B} = G = \frac{-g_m Z_L}{\left(1 + Z_L Y_{out}\right)}$$

the current through C_{BC} is:

$$I_{CBC} = j\omega C_{BC}\left(V_B - V_C\right) = j\omega C_{BC}\left(1 - G\right)V_B = j\omega C_{BC}\left(1/G - 1\right)V_C$$

We can account for C_{BC} by introducing an additional input "Miller" admittance:

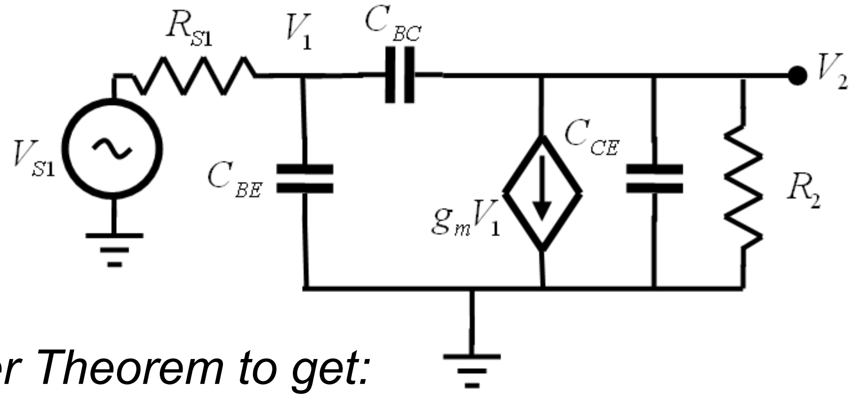
$$Y_{M,in} = j\omega C_{BC}\left(1 - G\right)$$

and an additional output "Miller" admittance:

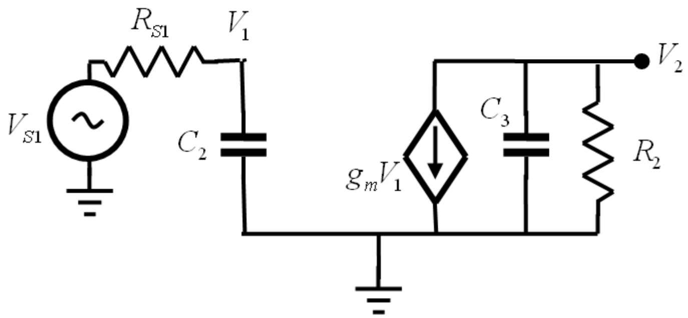
$$Y_{M,out} = j\omega C_{BC}\left(1 - 1/G\right)$$

Frequency response of the CE amplifier

V_{S1} and R_{S1} is the Thevenin equivalent of R_S V_S and R_π



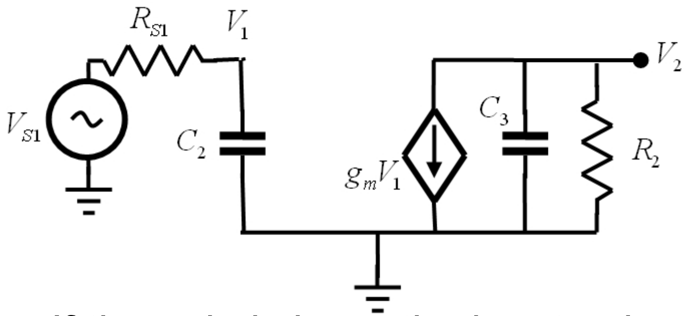
Use the Miller Theorem to get:



with $C_2 = C_{BE} + (1+G)C_{BC} \simeq (1+G)C_{BC}$ and $C_3 = C_{CE} + (1+1/G)C_{BC} \simeq C_{BC}$

The low requency gain from V_1 to V_2 is: $G = -g_m R_2 \simeq -g_m Z_L$

Frequency response of the CE amplifier (2)



the gain of this amplifier is:

$$\frac{dV_2}{dV_{S1}} = \frac{-g_m}{1+sR_{S1}C_2}\frac{R_2}{1+sR_2C_3} = \frac{G}{\underbrace{(1+sR_{S1}C_2)}_{\text{input pole}}\underbrace{(1+sR_2C_3)}_{\text{output pole}}}$$

If the gain is large the input pole can be at a much lower frequency than the output pole:

$$\tau_{IN} = R_{S1}C_2 = (R_\pi // R_S)(C_{BE} + (1+G)C_{BC}) \simeq R_S(1+G)C_{BC} \quad (\text{since } R_S \ll R_\pi)$$

$$\tau_{OUT} = R_2C_3 = (C_{BC}(1+1/G) + C_{CE})(R_{CE} // Z_L) \simeq C_{BC}Z_L \quad (\text{since } C_{BC} \gg C_{CE})$$

if $\tau_{in} > \tau_{out}$ *The gain-bandwidth product of this amplifier is:*

$$\left.\begin{aligned} |G| &= g_m R_2 \simeq g_m Z_L \\ BW &= \frac{1}{2\pi R_{S1}C_2} \simeq \frac{1}{2\pi R_S C_2} \end{aligned}\right\} \Rightarrow GBW = \frac{g_m R_2}{2\pi R_S C_2}. \quad \lim_{|G|\to\infty} GBW \simeq \frac{G}{2\pi R_S C_{BC}(1+G)} \simeq \frac{1}{2\pi R_S C_{BC}} \; .$$

This is independent of G!

The tiny C_{BC} and the source impedance can define the gain-bandwidth product of the CE amp.

This suggests that feedback is at work!

Note that the input pole will be at a lower frequency than the output pole if

$$|G| R_{S1} > R_2 \Rightarrow g_m R_{S1} R_2 > R_2 \Rightarrow g_m R_S = \beta R_S / R_\pi > 1$$

Importance of the gain-bandwidth product

the gain of the CE amplifier is:

$$\frac{dV_2}{dV_{S1}} = \frac{-g_m}{1+sR_{S1}C_2}\frac{R_2}{1+sR_2C_3} = \frac{G}{\underbrace{\left(1+sR_{S1}C_2\right)}_{\text{input pole}}\underbrace{\left(1+sR_2C_3\right)}_{\text{output pole}}}$$

One of the two pole frequencies is usually much lower than the other.
We then say that we have a ***dominant pole amplifier*** *whose frequency response is:*

$$A_v(s) = \frac{G_0}{1+s\tau} = \frac{G_0}{1+j\omega\tau}$$ for a DC gain G_0 and a bandwidth $\omega_B = 1/\tau$

We have already seen that in the limit of large G_0 **the Gain-Bandwidth product**:

$$G_B = G_0\omega_B = G_0/\tau$$

is independent of the amplifier gain.

We will see later in the course that this is a very general result which **applies to all one pole feedback amplifiers** regardless of how they are constructed.