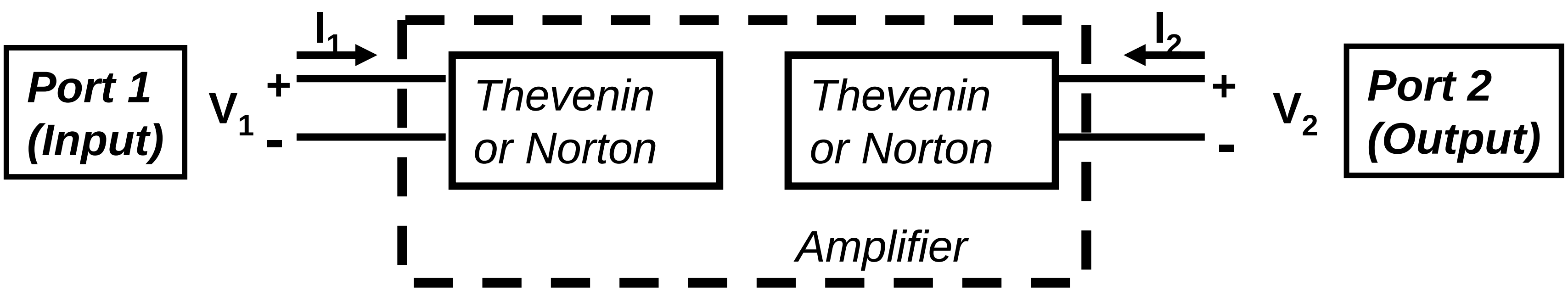
The dreaded “2-port parameters”

Aims:

- *To generalise the Thevenin an Norton Theorems to devices with 3 terminals*
- *Develop efficient computational tools to handle feedback connections of non ideal devices*

Generalised Thevenin + Norton Theorems

- Amplifiers, filters etc have input and output “ports” (pairs of terminals)
- By convention:
  - *Port numbering is left to right: left is port 1, right port 2*
  - *Output port is to the right of input port. e.g. output is port 2.*
  - *Current is considered positive flowing into the positive terminal of port*
  - *The two negative terminals are usually considered connected together*
- There is both a voltage and a current at each port
  - *We are free to represent each port as a Thevenin or Norton*
- Since the output depends on the input (and vice-versa!) any Thevenin or Norton sources we use must be dependent sources.
- General form of amplifier or filter:



How to construct a 2-port model

- Decide the representation (Thevenin or Norton) for each of:
  - *Input port*
  - *Output port*
- Remember the I-V relations for Thevenin and Norton Circuits:
  - *A Thevenin circuit has I as the independent variable and V as the dependent variable:* $V = V_T + I\,R_T$
  - *A Norton circuit has V as the independent variable and I as the dependent variable:* $I = I_N + V\,G_N$
- The source of one port is controlled by the independent variable of the other port.
- Write the I-V relations.
- In what follows we examine the 4 obvious choices of building a 2-port model.

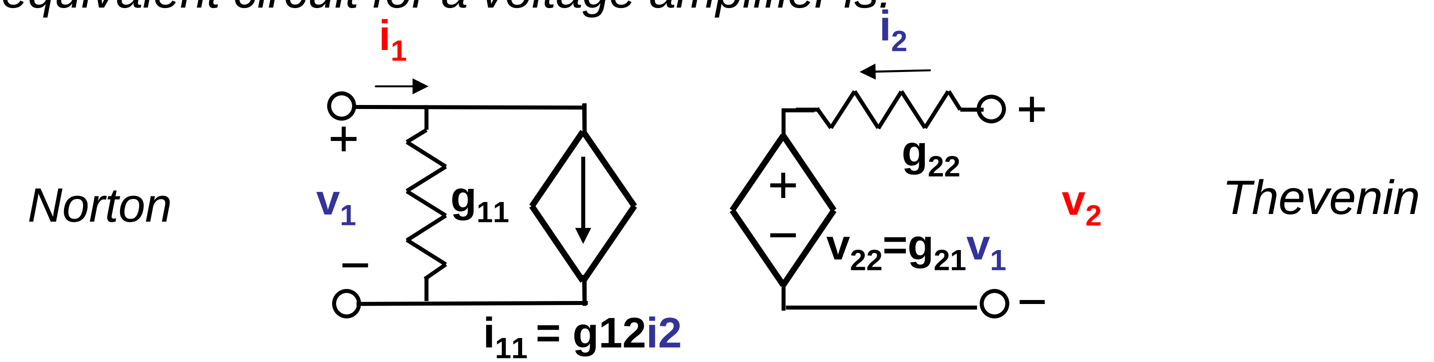
Modelling the voltage amplifier

G parameters or "reverse hybrid parameters"

To represent a voltage amplifier we require that a voltage driving port 1 results into a voltage developed on port 2.
This dictates the choice:

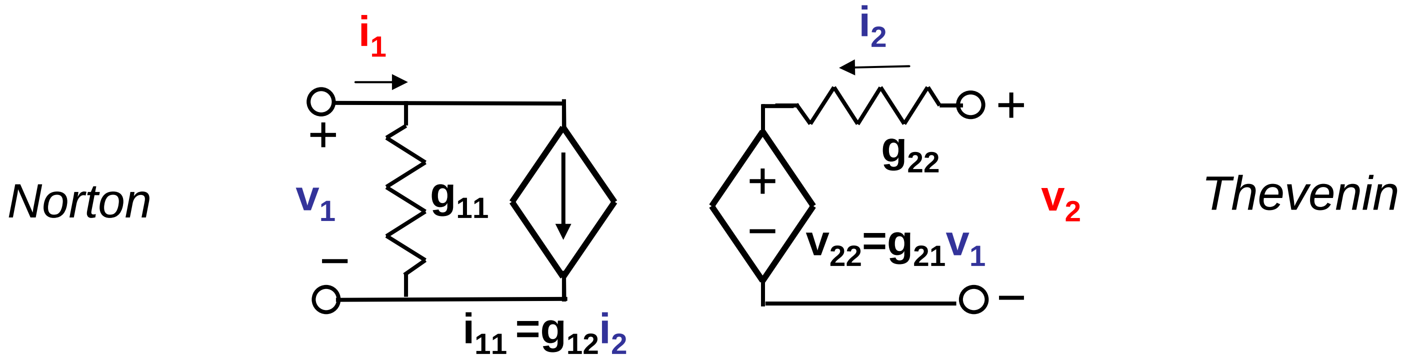
- *Voltage input for port 1,* ➔ *Norton representation for port 1.*
- *Voltage output for port 2,* ➔ *Thevenin representation for port 2.*

An equivalent circuit for a voltage amplifier is:



- *We have indicated the Thevenin source of port 2 as a voltage controlled voltage source $v_{22}=g_{21}v_1$. This represents the main function of the voltage amplifier.*
- *The Norton source of port 1 is also a controlled source, a CCCS, namely $i_{11}=g_{12}i_2$. The amplifier may have a non-zero reverse current gain.*
- *If the reverse current gain g_{12} is zero, the amplifier is called* **Unilateral**
- *The voltage input terminal has a finite admittance g_{11}.*
- *The voltage output terminal has a finite impedance g_{22}.*

Modelling the voltage amplifier (2)



We can write the I-V equations for each of the 2 terminals:

$$\left.\begin{aligned} i_1 &= g_{11}v_1 + i_{11} = g_{11}v_1 + g_{12}i_2 \\ v_2 &= v_{22} + g_{22}i_2 = g_{21}v_1 + g_{22}i_2 \end{aligned}\right\} \Rightarrow$$

$$\begin{bmatrix} i_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} v_1 \\ i_2 \end{bmatrix} \Rightarrow \begin{bmatrix} i_1 \\ v_2 \end{bmatrix} = \mathbf{G} \begin{bmatrix} v_1 \\ i_2 \end{bmatrix}$$

Terminology:

g_{11}: Input admittance

g_{12}: Reverse current gain

g_{21}: Voltage gain

g_{22}: Output impedance

A few final observations, without proof:

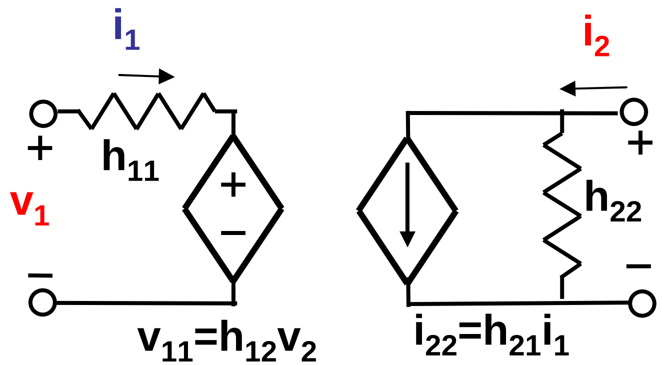
- *The forward signal path exhibits more than unity **power** gain.*
- *If connected in the reverse sense (input to port 2 and output from port 1) the amplifier will exhibit less than unity **power** gain.*

Modelling a current amplifier

H parameters or the "hybrid parameters"

A current amplifier has current input at port 1 and current and output at port 2. The representation choice is the inverse of that of a voltage amplifier:

Thevenin



Norton

$$\left.\begin{aligned} v_1 &= h_{11} i_1 + v_{11} = h_{11} i_1 + h_{12} v_2 \\ i_2 &= i_{22} + h_{22} v_2 = h_{21} i_1 + h_{22} v_2 \end{aligned}\right\} \Rightarrow$$

$$\begin{bmatrix} v_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} i_1 \\ v_2 \end{bmatrix} = \mathbf{H} \begin{bmatrix} i_1 \\ v_2 \end{bmatrix}$$

Terminology

h_{11}: Input impedance

h_{12}: Reverse voltage gain

h_{21}: Current gain

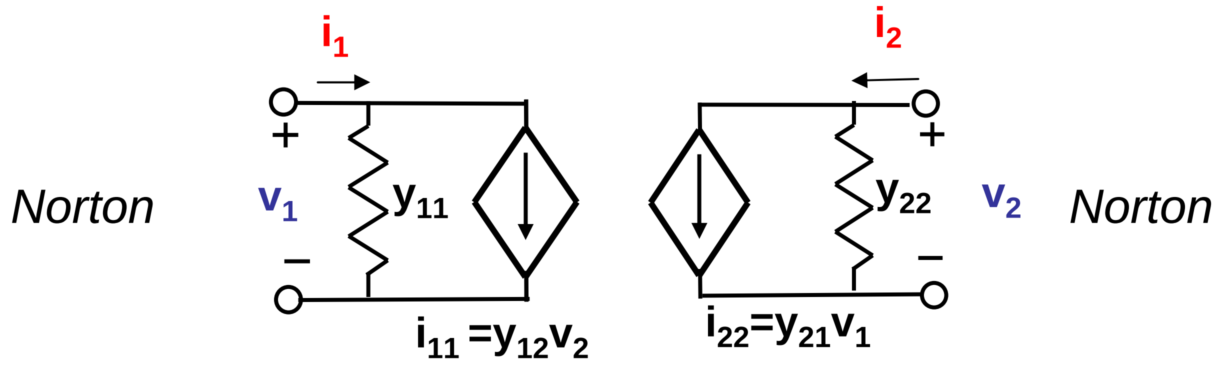
h_{22}: Output admittance

- *A current amplifier exhibits a reverse voltage gain (h_{12})!*
- *The port-reversed amplifier is a voltage "amplifier"*
- *The reverse signal path exhibits less than unity* ***power*** *gain.*
- *If the reverse voltage gain h_{12} is zero, the amplifier is called* **Unilateral**

The FET: a transconductance amplifier

Y parameters or the "short circuit parameters"

This is a generalisation of an admittance. Both input variables are voltages.



Formal description

y_{11}: Input admittance

y_{12}: Reverse admittance gain

y_{21}: trans-admittance (gain)

y_{22}: Output admittance

$$\left.\begin{aligned} i_1 &= y_{11}v_1 + i_{11} = y_{11}v_1 + y_{12}v_2 \\ i_2 &= i_{22} + y_{22}v_2 = y_{21}v_1 + y_{22}v_2 \end{aligned}\right\} \Rightarrow$$

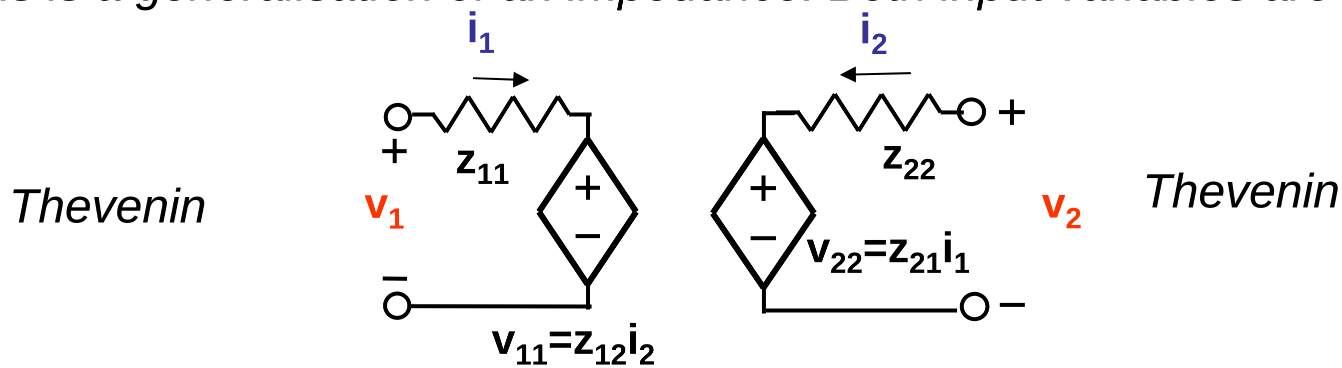
$$\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \mathbf{Y} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

- *A transconductance amplifier exhibits a reverse transconductance gain!*
- *The port-reversed amplifier is also a transconductance "amplifier"*
- *The reverse signal path exhibits less than unity **power** gain.*
- *If the reverse gain y_{12} is zero, the amplifier is called* **Unilateral**

The transresistance amplifier

Z parameters or the "open circuit parameters"

This is a generalisation of an impedance. Both input variables are currents.



Formal description

z_{11}: *Input impedance*

z_{12}: *Reverse impedance gain*

z_{21}: *transimpedance gain*

z_{22}: *Output impedance*

$$\left.\begin{aligned} v_1 &= z_{11}i_1 + v_{11} = z_{11}i_1 + z_{12}i_2 \\ v_2 &= v_{22} + z_{22}i_2 = z_{21}i_1 + z_{22}i_2 \end{aligned}\right\} \Rightarrow$$

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \mathbf{Z} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}$$

- *A transresistance (also called a transimpedance) amplifier exhibits a reverse transimpedance gain!*
- *The port-reversed amplifier is also a transresistance "amplifier"*
- *The reverse signal path exhibits less than unity* ***power*** *gain.*
- *If the reverse gain z_{12} is zero, the amplifier is called* **Unilateral**

Determining values of 2-port parameters

We can determine 2-port parameters from the definitions. For example, the G parameter description states that:

$$i_1 = g_{11}v_1 + g_{12}i_2$$
$$v_2 = g_{21}v_1 + g_{22}i_2$$

Since these are small signal voltages and currents, these relations imply that the g parameters are partial derivatives:

$$g_{11} = \left.\frac{\partial i_1}{\partial v_1}\right|_{i_2=0} \qquad g_{12} = \left.\frac{\partial i_1}{\partial i_2}\right|_{v_1=0}$$
$$g_{21} = \left.\frac{\partial v_2}{\partial v_1}\right|_{i_2=0} \qquad g_{22} = \left.\frac{\partial v_2}{\partial i_2}\right|_{v_1=0}$$

The voltage boundary condition v_1=0 means that v_1 is kept constant, by connecting to a constant voltage source.
Similarly, the current boundary condition i_2=0, requires that the variation of the current flowing into a terminal is zero, i.e. the current originates from a constant current source.

Input and output impedance of an amplifier

To connect a 2-port to other circuits we need to know its input and output impedance. Although it would appear these are the 11 and 22 entries of the appropriate parameter matrix this is not necessarily the case.

Since the choice of parameter representation is arbitrary, lets consider 2 different representations for a circuit, the Z and Y parameters. It is easy to confirm that matrix Z must be the inverse of matrix Y.

This means that $$Z_{11} = \left.\frac{\partial v_1}{\partial i_1}\right|_{i_2=0} = \left(\overleftrightarrow{Y}^{-1}\right)_{11} = \frac{Y_{22}}{Y_{11}Y_{22} - Y_{12}Y_{21}} \neq \frac{1}{Y_{11}} = \left(\left.\frac{\partial i_1}{\partial v_1}\right|_{v_2=0}\right)^{-1}$$

What is wrong here? Z_{11} is determined by a fictitious experiment in which port 2 is driven by a current source (i_2=0) while Y_{11} is the result of a fictitious experiment in which port 2 is driven by a voltage source (v_2=0)

Clearly the input impedance depends on the load connected to the output.

Similarly, the output impedance depends on what is connected to the input.

Observe that the terminal impedance does not depend on what is connected to the other port if the network is unilateral, i.e. if Y_{12}=0 ➔ $Z_{11}=1/Y_{11}$.

Input and output impedance of an amplifier (2)

We showed that the input impedance of an amplifier depends on whether the output is connected to a zero or infinite thevenin impedance. This allows us to prove by "*reductio ad absurdum":*

the input impedance depends on the load impedance connected to the output

(if it did not, then the input impedance would be the same for the output open or shorted).

The exact value of the **loaded** input impedance can be found from the definition of the parameters, and ohm's law on the load R_L connected to the output:

$$\left.\begin{aligned} v_1 &= Z_{11}i_1 + Z_{12}i_2 \\ \left.\begin{aligned} v_2 &= Z_{21}i_1 + Z_{22}i_2 \\ v_2 &= -i_2 R_L \end{aligned}\right\} &\Rightarrow -i_2 R_L = Z_{21}i_1 + Z_{22}i_2 \Rightarrow i_2 = \frac{-Z_{21}i_1}{R_L + Z_{22}} \end{aligned}\right\} \Rightarrow$$

$$Z_{in} = \frac{v_1}{i_1} = Z_{11} - \frac{Z_{12}Z_{21}}{R_L + Z_{22}}$$

The input impedance will only equal Z_{11} if

- the amplifier is unilateral, or
- the output is open circuited so that the load resistance is infinite.

Conversion between amplifier representations

- Some are obvious matrix inversions from the matrix equations:

$$\mathbf{Z} = \mathbf{Y}^{-1}$$

$$\mathbf{G} = \mathbf{H}^{-1}$$

Recall that the inverse of a 2x2 matrix A is:

$$\mathbf{A}^{-1} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1} = \frac{1}{\Delta_a}\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix},$$

$$\Delta_a = a_{11}a_{22} - a_{21}a_{12}$$

- To go from pure (Z or Y) to/from hybrid (G or H) parameters, e.g. to express y in terms of h:
  - *write the definitions in terms of the known parameters (h in the example).*
  - *separate the independent/ dependent variables in the unknown (y in the example) representation. One equation will be trivial to obtain! Substitute this into the other eq.*

$$\left.\begin{aligned} & v_1 = h_{11}i_1 + h_{12}v_2 \Rightarrow i_1 = \frac{1}{h_{11}}v_1 - \frac{h_{12}}{h_{11}}v_2 = y_{11}v_1 + y_{12}v_2 \\ & i_2 = h_{21}i_1 + h_{22}v_2 \end{aligned}\right\} \Rightarrow$$

$$\Rightarrow i_2 = h_{21}\left(\frac{1}{h_{11}}v_1 - \frac{h_{12}}{h_{11}}v_2\right) + h_{22}v_2 \Rightarrow i_2 = \frac{h_{21}}{h_{11}}v_1 + \frac{h_{11}h_{22} - h_{21}h_{12}}{h_{11}}v_2 = y_{21}v_1 + y_{22}v_2$$

The equations for the two currents in terms of the voltages are the y-parameter equations.

Common features of 2-port representations

- x are the chosen inputs and y the chosen outputs. Then we can write

$$\left.\begin{aligned} y_1 &= ax_1 + bx_2 \\ y_2 &= cx_1 + dx_2 \end{aligned}\right\} \Rightarrow \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

- a is the input impedance or admittance (depending on its units)
- d is the output impedance or admittance (depending on its units)
- c is the forward gain
- b is the reverse gain. All 2-port networks may exhibit non-zero reverse gain!
- If b=0 we call the 2-port network unilateral
- If the network is meant to be an amplifier then its forward power gain is greater than unity, and its reverse power gain is less than unity.
- If a load is connected to port 2 the resistance seen into port 1 depends on the magnitude of the load, unless the network is unilateral.
- The output impedance of a network depends on the impedance driving its input unless the network is unilateral.

Amplifier representation choice

- It does not matter which representation is used, use most convenient!
- Sensible choice depends on <u>what is connected to the amplifier</u>

- If $Z_{source} << Z_{in}$ use voltage input representation:
  - *Norton input half circuit: voltage (G) or transconductance (Y)*
- If $Z_{source} >> Z_{in}$ use current input
  - *Thevenin input half circuit: current (H) or transimpedance (Z)*
- If $Z_{load} >> Z_{out}$ use voltage output representation:
  - *Thevenin output half circuit: voltage (G) or transimpedance (Z)*
- If $Z_{load} << Z_{out}$ use current output representation:
  - *Norton input half circuit: current (H) or transconductance (Y)*

Amplifiers: modelling summary

Name / Representation	Parameters	Input	Output	Forward gain	Reverse gain
Voltage	G	Norton	Thevenin	Voltage	Current
Current	H	Thevenin	Norton	Current	Voltage
Transconductance	Y	Norton	Norton	Admittance	Admittance
Transimpedance	Z	Thevenin	Thevenin	Impedance	Impedance

Name / Representation	Input	Output	Ideal form	Terminal impedance			
				Ideal		Real	
				Input	Output	Input	Output
Voltage	V	V	VCVS	∞	0	High	Low
Current	I	I	CCCS	0	∞	Low	High
Transconductance	V	I	VCCS	∞	∞	High	High
Transimpedance	I	V	CCVS	0	0	Low	Low

***Notes**:*

1. Choice of representation is **arbitrary**
2. Representation emphasises the intended function
3. *Can convert one representation into any other by Thevenin ←→ Norton transforms*

formal analysis of common terminal change
example: how to get the CB Y matrix from the CE Y matrix

- ***The CB terminal voltages and currents, and their correspondence to CE voltages and currents are:***

$$v_{1b} = v_{EB} = -v_{BE} = -v_{1e}$$

$$v_{2b} = v_{CB} = v_{CE} - v_{BE} = v_{2e} - v_{1e}$$

$$i_{1b} = i_E = -i_B - i_C = -i_{1e} - i_{2e}$$

$$i_{2b} = i_C = i_{2e}$$

the admittance equations in the CB and CE configurations must be equivalent:

$$i_{1b} = -i_{1e} - i_{2e} = \underbrace{-y_{ie}v_{1e} - y_{re}v_{2e} - y_{fe}v_{1e} - y_{oe}v_{2e}}_{CE} = \underbrace{y_{ib}v_{1b} + y_{rb}v_{2b}}_{CB} = -y_{ib}v_{1e} + y_{rb}\left(v_{2e} - v_{1e}\right) \Rightarrow$$

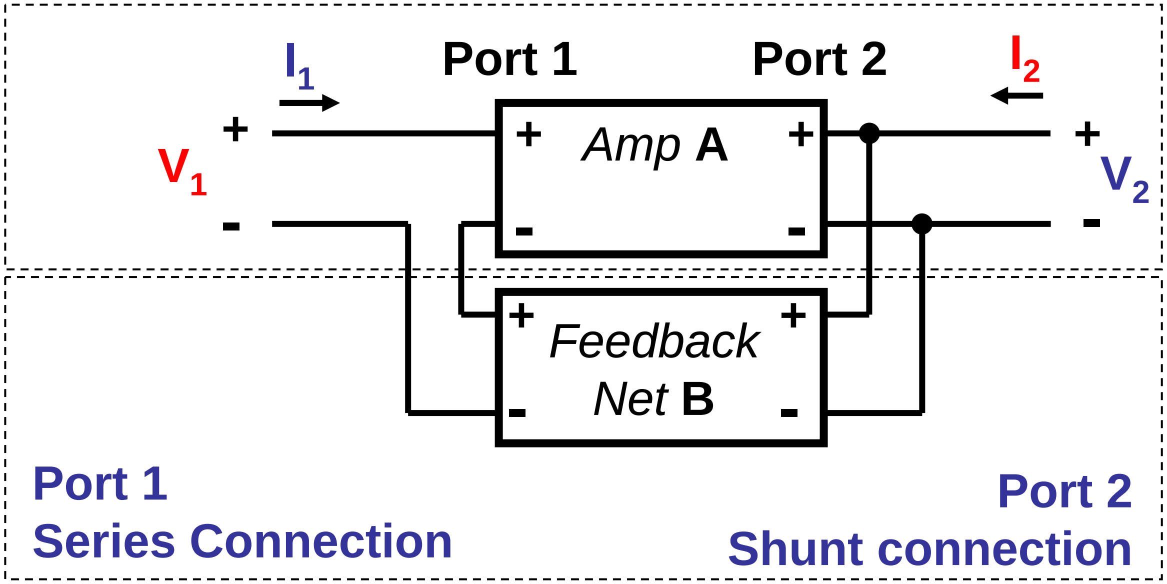
$$\Rightarrow \left\{\begin{matrix} y_{ib} + y_{rb} = y_{ie} + y_{fe} \\ y_{rb} = -\left(y_{re} + y_{oe}\right) \end{matrix}\right\} \Rightarrow \left\{\begin{matrix} y_{rb} = -\left(y_{re} + y_{oe}\right) \\ y_{ib} = y_{ie} + y_{fe} + y_{re} + y_{oe} \end{matrix}\right.$$

$$i_{2b} = i_C = i_{2e} = \underbrace{y_{fe}v_{1e} + y_{oe}v_{2e} = y_{fb}v_{1b} + y_{ob}v_{2b}}_{CE} = \underbrace{y_{fb}v_{1b} + y_{ob}v_{2b}}_{CB} = -y_{fb}v_{1e} + y_{ob}\left(v_{2e} - v_{1e}\right) \Rightarrow$$

$$\Rightarrow \left\{\begin{matrix} y_{fb} + y_{ob} = -y_{fe} \\ y_{ob} = y_{oe} \end{matrix}\right\} \Rightarrow \left\{\begin{matrix} y_{ob} = y_{oe} \\ y_{fb} = -\left(y_{fe} + y_{oe}\right) \end{matrix}\right.$$

Exact treatment of the Series-Shunt connection:

Add H parameters



Function of feedback net:
Measure output V
Correct (mix) input V
i.e. it improves a G-amp

Shared electrical variables:

I_1 , V_2

The feedback network is functionally a voltage amplifier from Port2 to Port1
Electrically both networks share the electrical variables I_1 and V_2.
KVL on P1 and KCL on P2 give:

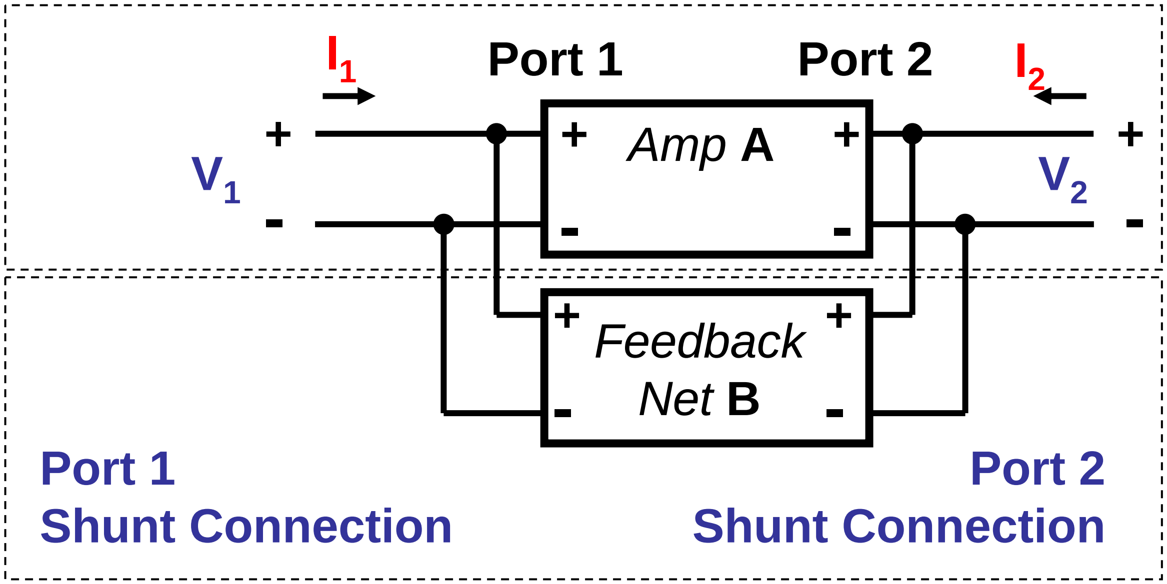
$$V_1 = V_{1A} + V_{1B} \ , I_2 = I_{2A} + I_{2B} \Rightarrow$$

$$\begin{bmatrix} V_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} V_{1A} \\ I_{2A} \end{bmatrix} + \begin{bmatrix} V_{1B} \\ I_{2B} \end{bmatrix} = \mathbf{H_A} \begin{bmatrix} I_{1A} \\ V_{2A} \end{bmatrix} + \mathbf{H}_B \begin{bmatrix} I_{1B} \\ V_{2B} \end{bmatrix} = \left(\mathbf{H_A} + \mathbf{H_B} \right) \begin{bmatrix} I_1 \\ V_2 \end{bmatrix} = \mathbf{H_{A+B}} \begin{bmatrix} I_1 \\ V_2 \end{bmatrix}$$

$$\text{since } I_{1A} = I_{1B} = I_1 \text{ and } V_{2A} = V_{2B} = V_2$$

Exact treatment of the Shunt-Shunt connection:

Add Y parameters



Function of feedback net:
Measure output V
Correct (mix) input I
i.e. it improves a Z-amp

Shared electrical variables:

V_1, V_2

The feedback network is functionally a transconductance amplifier from Port2 to Port1
Electrically the networks share V_1 and $V_{2.}$ Application of KCL on each port gives:

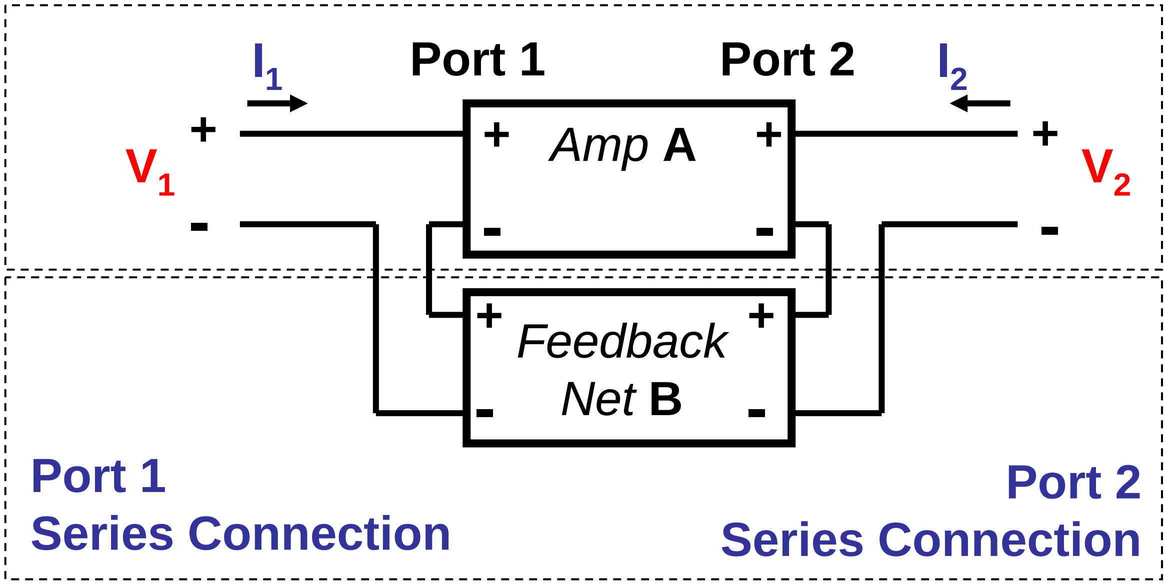
$$I_1 = I_{1A} + I_{1B} \;,\; I_2 = I_{2A} + I_{2B} \Rightarrow$$

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} I_{1A} \\ I_{2A} \end{bmatrix} + \begin{bmatrix} I_{1B} \\ I_{2B} \end{bmatrix} = \mathbf{Y_A} \begin{bmatrix} V_{1A} \\ V_{2A} \end{bmatrix} + \mathbf{Y_B} \begin{bmatrix} V_{1B} \\ V_{2B} \end{bmatrix} = \left(\mathbf{Y_A} + \mathbf{Y_B}\right) \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \mathbf{Y_{A+B}} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

$$\text{since } V_{1A} = V_{1B} = V_1 \text{ and } V_{2A} = V_{2B} = V_2$$

Exact treatment of the Series - Series connection:

Add Z parameters



Function of feedback net:
Measure output I
Correct (mix) input V
i.e. it improves a Y-amp

Shared electrical variables:

I_1 , I_2

The feedback network is functionally a transimpedance amplifier from Port2 → Port1
Electrically the two networks share input and output currents. Apply KVL on both ports:

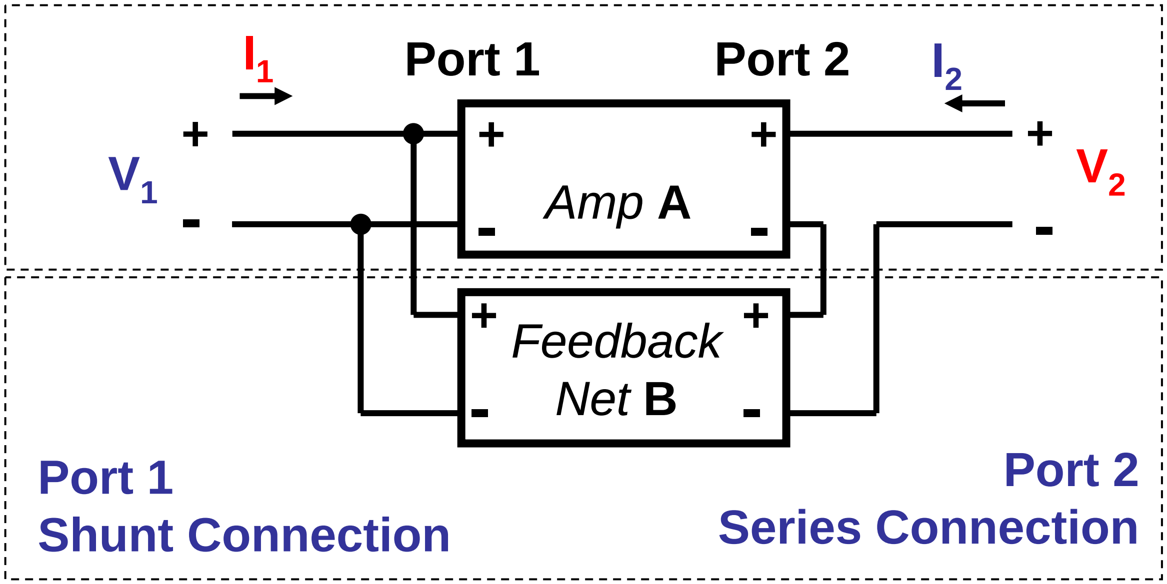
$$V_1 = V_{1A} + V_{1B} \text{ and } V_2 = V_{2A} + V_{2B} \Rightarrow$$

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} V_{1A} \\ V_{2A} \end{bmatrix} + \begin{bmatrix} V_{1B} \\ V_{2B} \end{bmatrix} = \mathbf{Z_A} \begin{bmatrix} I_{1A} \\ I_{2A} \end{bmatrix} + \mathbf{Z}_B \begin{bmatrix} I_{1B} \\ I_{2B} \end{bmatrix} = (\mathbf{Z_A} + \mathbf{Z_B}) \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \mathbf{Z_{A+B}} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

$$\text{since } I_{1A} = I_{1B} = I_1 \text{ and } I_{2A} = I_{2B} = I_2$$

Exact treatment of the Shunt - Series connection:

Add G parameters



Function of feedback net:
Measure output I
Correct (mix) input I
i.e. it improves an H-amp

Shared electrical variables:

V_1, I_2

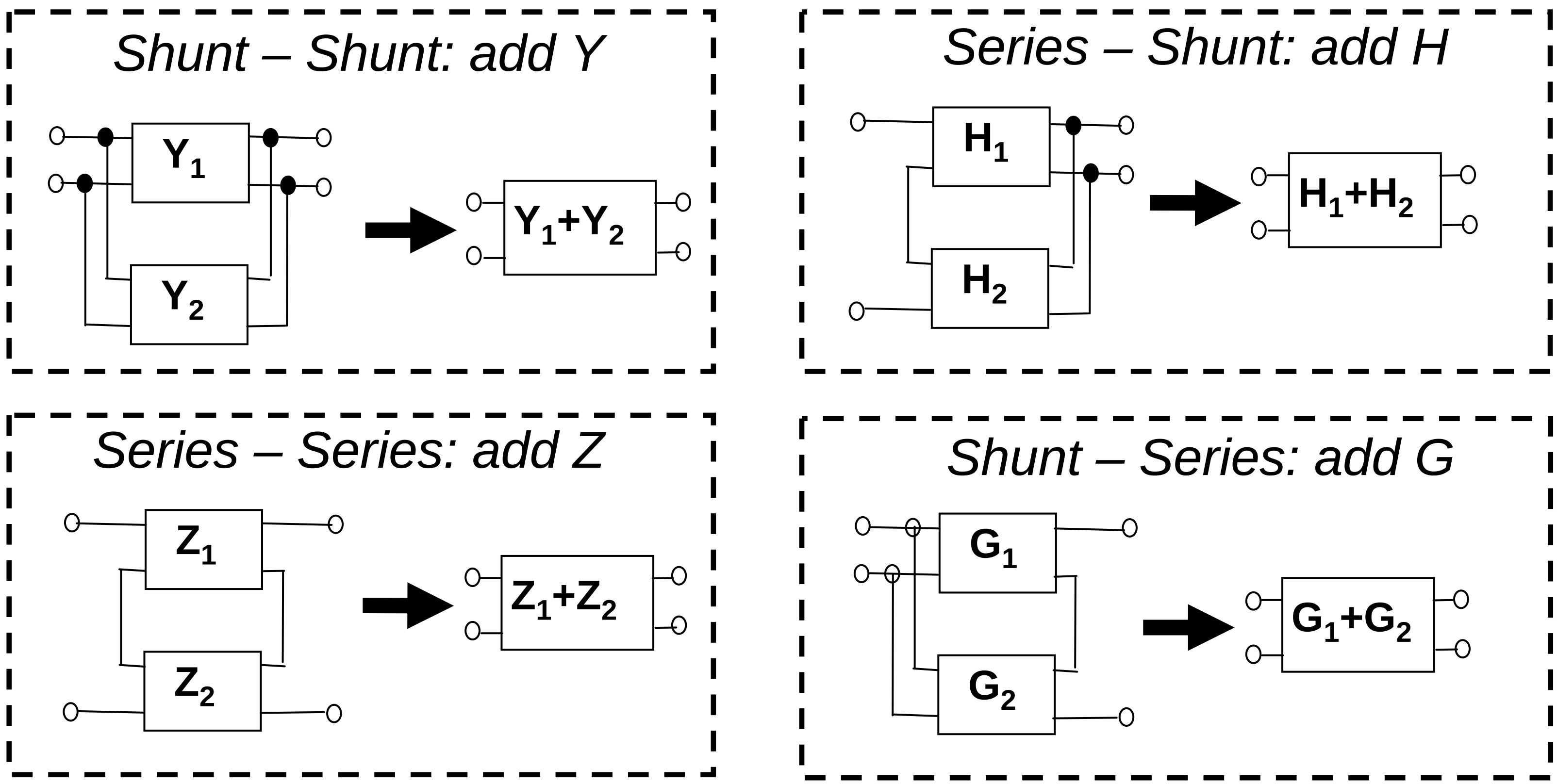
The feedback network is functionally a current amplifier from Port2 to Port1
Electrically the two networks share V_1 and I_2. KCL on P1 and KVL on P2 gives:

$$I_1 = I_{1A} + I_{1B} \text{ and } V_2 = V_{2A} + V_{2B} \Rightarrow$$

$$\begin{bmatrix} I_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} I_{1A} \\ V_{2A} \end{bmatrix} + \begin{bmatrix} I_{1B} \\ V_{2B} \end{bmatrix} = \mathbf{G_A} \begin{bmatrix} V_{1A} \\ I_{2A} \end{bmatrix} + \mathbf{G_B} \begin{bmatrix} V_{1B} \\ I_{2B} \end{bmatrix} = (\mathbf{G_A} + \mathbf{G_B}) \begin{bmatrix} V_1 \\ I_2 \end{bmatrix} = \mathbf{G_{A+B}} \begin{bmatrix} V_1 \\ I_2 \end{bmatrix}$$

$$\text{since } V_{1A} = V_{1B} = V_1 \text{ and } I_{2A} = I_{2B} = I_2$$

2-port network feedback connection rules



*These are **not** all the possible amplifier interconnections. We often need to cascade two amplifiers, i.e. drive the an amplifier with the output of another. We will encounter this connection when we study multi-stage amplifiers.*

2-port parameters: the "old" notation

The choice of common terminal introduces ambiguity in the 2-port parameter description of a transistor. This is resolved in the "old" (IRE) notation as follows:

- A 2 lowercase letter subscript is used instead of the numerical we have already seen.:
- The first letter is
  - *"i" for the Input Impedance or Admittance (the "11" parameter)*
  - *"f" for the Forward Gain (the "21" parameter)*
  - *"o" for the Output Impedance or Admittance (the "22" parameter)*
  - *"r" for the Reverse Gain (the "12" parameter)*
- The second letter specifies which of the device terminals is used as a common reference: Emitter ("e"), Base ("b") or Collector ("c").
- For field effect transistors the second letter is "s" (source), "g" (gate) or "d" (drain)
- Example: In the common emitter connection the "h" parameters are written:

h_{ie} input impedance (h_{11})

h_{re} reverse voltage gain (h_{12})

h_{fe} forward current gain (h_{21}) also known as β

h_{oe} output admittance (h_{22})

This notation in still used in data sheets!