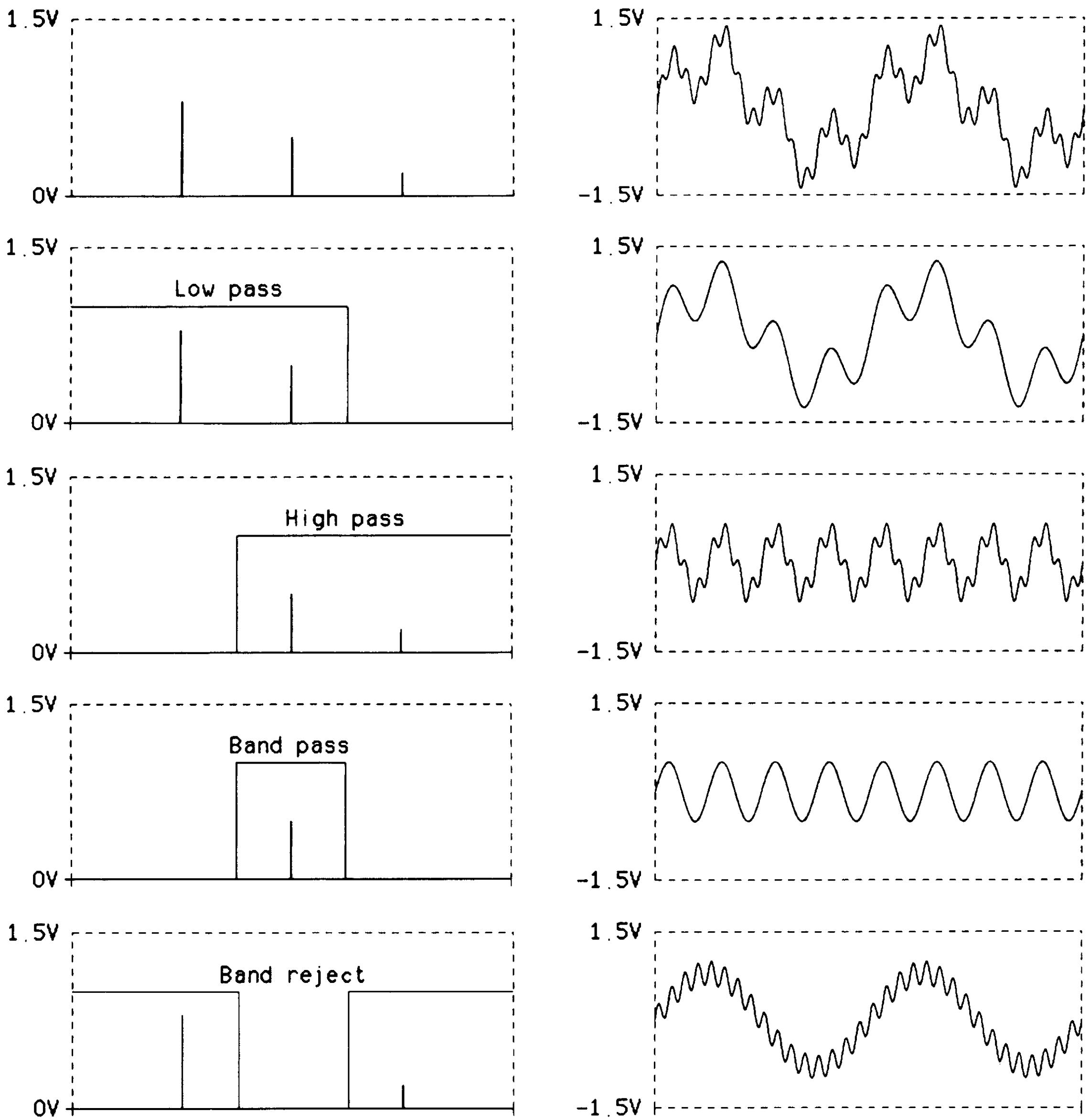
Active Filters

Motivation:

- Analyse filters
- Design low frequency filters without large capacitors
- Design filters without inductors
- Design electronically programmable filters

Some waveforms, to show the effect of filtering



Noisy sine

Low Pass

High Pass

Band Pass

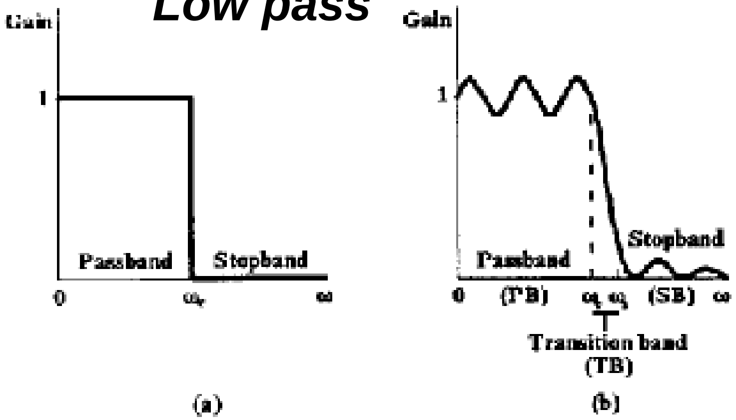
Band Reject

Frequency domain

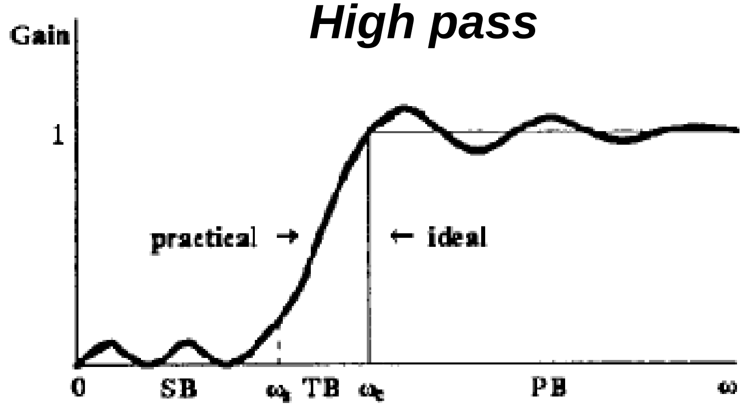
Time domain

Filter types

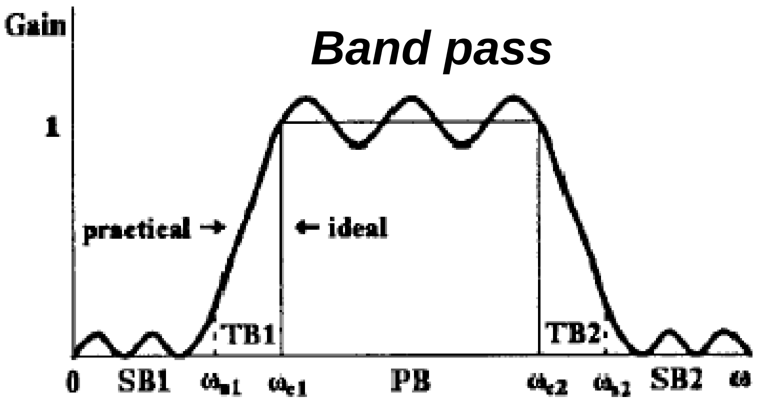
Low pass



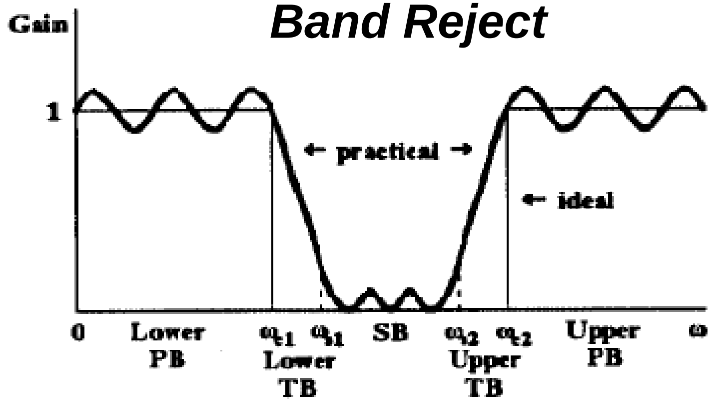
High pass



Band pass



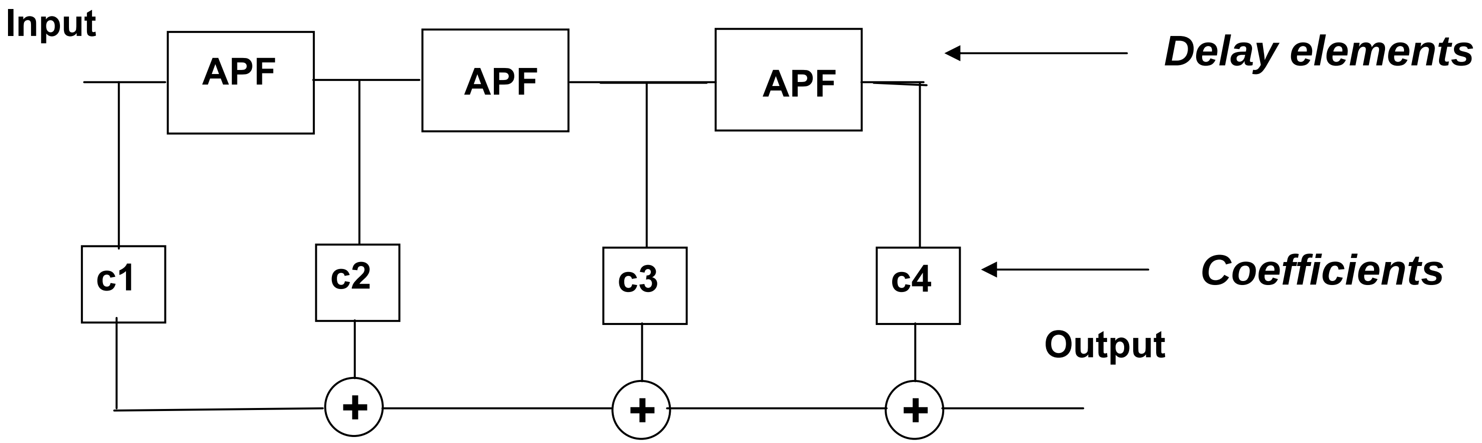
Band Reject



Observe that a real filter is not sharp, and its transmission is not constant!

All Pass Filters

- Filters do not only change magnitude of signal
- Filters alter phase as a function of frequency, i.e. introduce delays
- The derivative of phase is a time delay
- All pass filters delay signals without affecting their magnitude
- All pass filters can be used to synthesise other filters:



- All pass filter based analogue filters are similar to the digital filters encountered in Digital Signal Processing

The transfer function

- The transfer function is the Fourier transform of the impulse response
- Filters we can make have a rational transfer function: the transfer function is is a ratio of two polynomials with real coefficients.

  (strictly speaking this is called the "Padé approximation": it states that any real function can be approximated by a rational function. The higher the degree of the polynomials the closer the approximation can be made)

  The notation is s=jω. The signals assumed to be sinusoid: $V = V_0 e^{j\omega t+\phi}$

$$H(s) = \frac{P_n(s)}{Q_n(s)} = \frac{\sum_{k=0}^{n} a_k s^k}{\sum_{k=0}^{m} b_k s^k} = \frac{a_n (s-z_1)(s-z_2)\cdots(s-z_n)}{b_m (s-p_1)(s-p_2)\cdots(s-p_m)}$$

- The roots z_k of the numerator polynomial are called the "zeroes" of H
- The roots p_k of the denominator polynomial are called the "poles" of H
- The pole positions on the complex frequency plane entirely determine the filter properties.
- Note that since s=jω the denominator is seldom zero

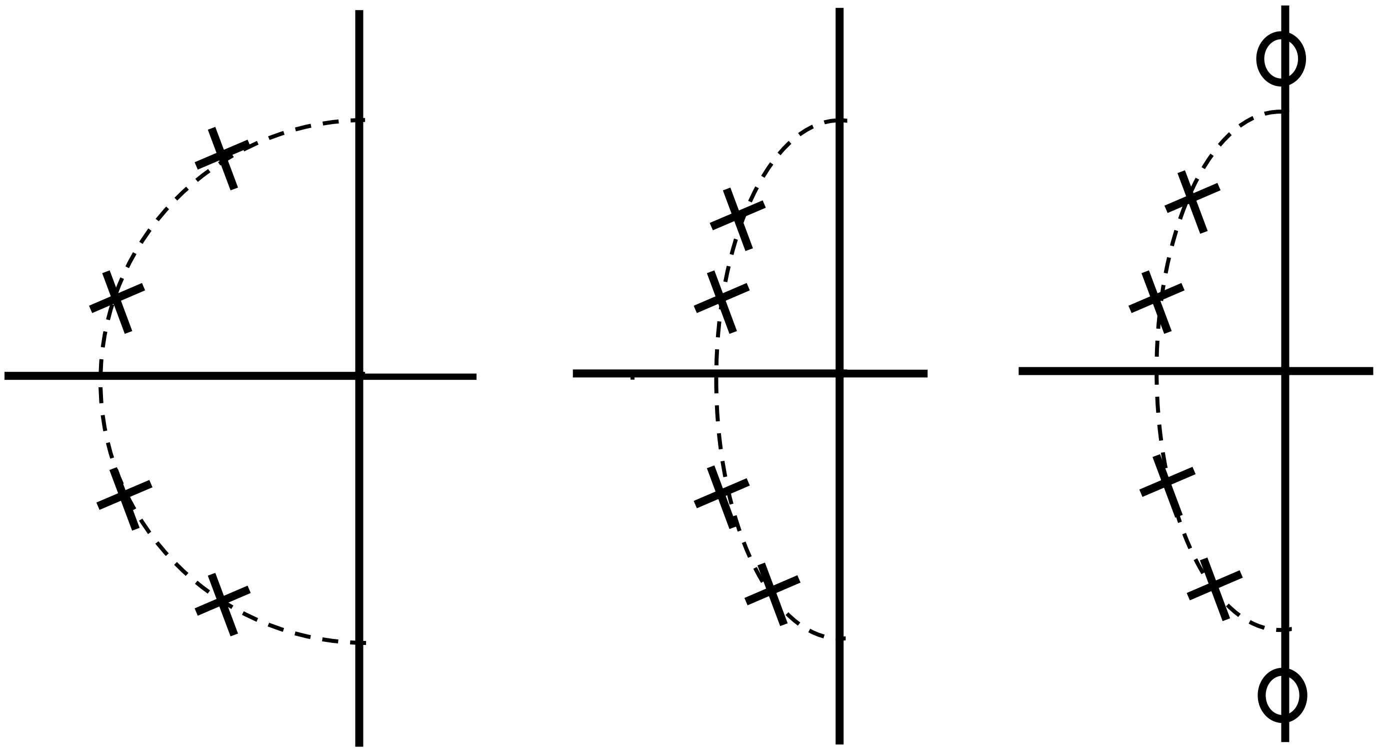
Families of filters

- Filters are classified into different families according to how the passband, stop band, transition region and group delay look like.
- Most filters you are likely to encounter have a low pass power transfer function of the form :

$$H(s)H^*(s) = \frac{1}{1+\varepsilon^2 P_n^2(s)}$$

- P_n is a suitable polynomial, or a polynomial approximation to some desired function. P_n are tabulated in reference books.
- Some common filter families (determined by P_n,) are:
  - *Butterworth. Maximally flat pass-band, slow transition to stop band*
  - *Chebyshev: Fast transition at the cost of pass-band ripple*
  - *Inverse Chebyshev: Fast transition at the cost of stop-band ripple*
  - *Elliptic: Fastest transition at the cost of ripple everywhere*
  - *Bessel: Maximally flat group delay (almost linear dependence of phase on frequency)*
- HPF, BPF, BRF, APF can be derived from a low pass prototype (next)
- Note that a fast passband - stopband transition results in a large variation of delay with frequency, i.e. unsuitable for digital signals!

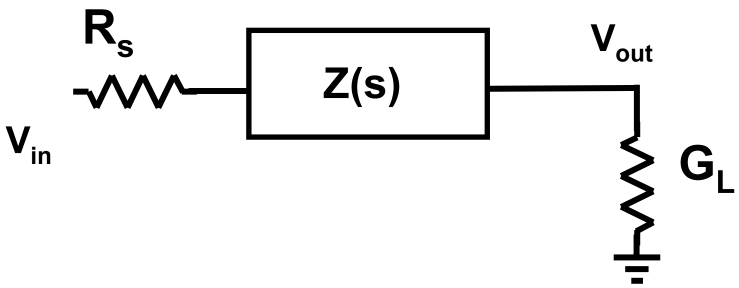
Pole-zero plots of Low Pass Filters



Pole locations determine filter response. The closer poles are to the imaginary axis the steepest the transition from passband to stopband.
a: Butterworth: poles on a circle
b: Chebyshev: Poles on an ellipse (sharper)
c: Elliptic: Like Chebyshev, plus zeroes on the imaginary axis (sharpest)

Passive filter synthesis

- Write the desired transfer function.
- Find Z(s) so that the following voltage divider is equal to the transfer function.

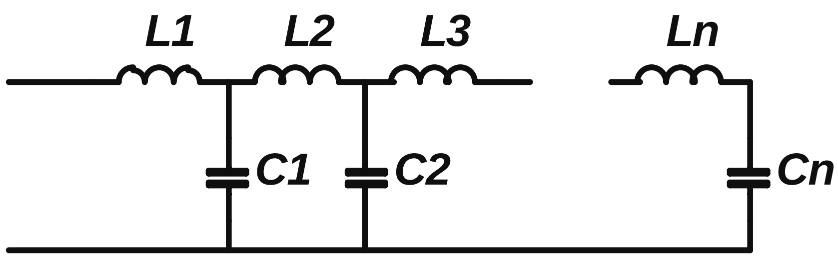


$$H(s) = \frac{v_{out}}{v_{in}} = \frac{1}{1 + \left(Z(s) + R_s\right)G_L} \Rightarrow Z(s) = \frac{1}{G_L}\left[\frac{1}{H(s)} - \left(1 + R_s G_L\right)\right]$$

- Use R,L,C to implement Z(s);
- Rs and Y_L are assumed known, usually real. The ideal cases Rs=0, Y_L=0 are trivial
- If Rs and Ys are not real we can add and subtract their imaginary parts from Z(s)
- There are many ways to make Z(s)
- We prefer “canonical forms”, which use least number of components
- We commonly use “Cauer forms” which are canonical ladder networks.

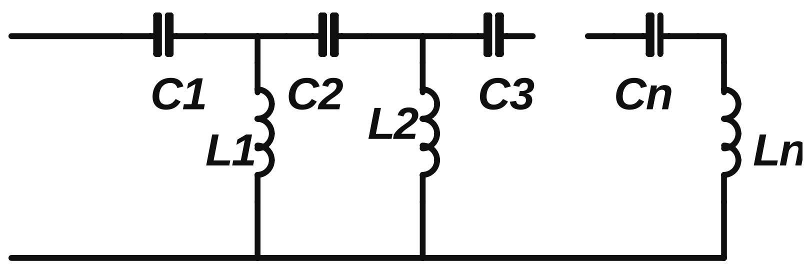
Cauer forms

First Cauer form



(a)

Second Cauer form



(b)

Cauer forms are derived by a continued fraction expansion of Z(s):

$$Z_{in} = sL_1 + \cfrac{1}{sC_1 + \cfrac{1}{sL_2 + \cfrac{1}{\cdots}}}$$

For the circuit on the left

Or, we can start from the Z-function:

$$\frac{s^3 + 2s}{s^2 + 1} = s + \frac{s}{s^2 + 1} = s + \cfrac{1}{s + \cfrac{1}{s}}$$

2^{nd} order filter transfer functions: Review

Second order filter transfer functions are all of the following form:

$$H(s) = H_0 \frac{C(s/\omega_0)^2 + 2B\zeta s/\omega_0 + A}{(s/\omega_0)^2 + 2\zeta s/\omega_0 + 1}\ ,\ \ Q = \frac{1}{2\zeta}$$

H_0 is the overall amplitude, ω_0 the break (or peak) frequency, and ζ the damping factor

ζ is related to the quality factor Q by: $Q=1/2\zeta$

The 3dB bandwidth of an underdamped 2^{nd} order filter is approx 1/Q times the peak frequency.

The coefficients A, B, C determine the function of the filter: →

Function	A	B	C
Low Pass	1	0	0
High Pass	0	0	1
Band Pass	0	1	0
Band Stop	1	0	1
All Pass	1	-1	1

2^{nd} order filters are useful: we can always decompose higher order filters to a cascade of 2^{nd} order filters!

Filters solve differential equations

Consider the ODE:

$$\left(\frac{1}{\omega_n^2}\frac{d^2}{dt^2}+\frac{2\zeta}{\omega_n}\frac{d}{dt}+1\right)y(t)=H_0\left(\frac{C}{\omega_n^2}\frac{d^2}{dt^2}+B\frac{2\zeta}{\omega_n}\frac{d}{dt}+A\right)x(t)$$

Substitute:

$$x=X(\omega)e^{j\omega t}=X(s)e^{st}\ ,\ y=Y(\omega)e^{j\omega t}=Y(s)e^{st}$$

To get:

$$H(s)=\frac{Y(s)}{X(s)}=H_0\frac{C(s/\omega_n)^2+2B\zeta s/\omega_n+A}{(s/\omega_n)^2+2\zeta s/\omega_n+1}$$

This is the transfer function of a 2nd order filter. It follows that the filter solves the ODE.

The impulse responses (IR) of lowpass, bandpass and highpass filters are related*:

- **The IR of the BP is proportional to the time derivative of the IR of the LP**
- **The IR of the HP is proportional to the time derivative of the IR of the BP**
- **It follows that a loop of 2 integrators can implement any 2nd order filter. Such a loop is called a “biquad”.**

*** (remember that H(s) is the Laplace transform of the impulse response)**

Filter transformations: LP←→HP

From a 2nd order low pass filter we can get a 2nd order high pass filter:

let $q = j\omega / \omega_n$ then for a 2nd order LPF:

$$H_{LP}(q) = \frac{H_0}{q^2 + 2\zeta q + 1}$$

$$H_{LP}(1/q) = \frac{H_0 q^2}{1 + 2\zeta q + q^2} = H_{HP}(q)$$

If the components of a filter are replaced so that any impedance dependence on ω is replaced by a similar dependence on 1/ω the filter changes from low pass to high pass
In practice we replace C with L and L with C so that:

$$\omega_n C = \frac{1}{\omega_n L}$$

The same transformation generates a low pass filter from a high pass filter.

Filter transformations LP←→BP

From a 1st order low pass filter we can get a 2nd order band pass filter:

let $q = j\omega / \omega_n$ then the transfer function of a 1^{st} order LPF is:

$$H(q) = \frac{H_0}{a+q}$$

$$H_1(q+1/q) = \frac{H_0}{a+(q+1/q)} = \frac{H_0 q}{q^2+aq+1} = H_{2BP}(q)$$

In practice we replace the low pass elements, following the following recipe:

- **all capacitors with parallel LC circuits, (open at resonance) and**
- **all inductors with series LC circuits (short at resonance)**

$$\omega_n C = \frac{1}{\omega_n L}$$

ω_n is the centre frequency of the filter. The BPF has the same BW as the LPF

$$\delta f = 4\pi\zeta\omega_n = 2\pi\alpha\omega_n = f_{B,LPF}$$

To get a band reject filter replace in the low pass prototype:
C→series LC
L→ parallel LC

Filter design from prototypes

Tabulated filter prototypes are usually given for low pass filters, with break frequency 1 rad/s and load impedance 1 ohm

From a LP filter prototype to get a HP filter with the same break frequency by the mapping: f→1/f.

- **replace C with L and L with C**
- **component values so that new components have same Z as old.**
- **for a 1rad/s prototype this means C→1/L, L→1/C**

From a LPF we get a BPF of bandwidth equal to the low pass bandwidth by:

- **Replacing each L with series LC resonating at ω_n. L stays the same**
- **Replacing each C with parallel LC resonating at ω_n. C stays the same**
- **Choosing the undetermined components to resonate at the filter centre frequency product**

From a high pass ladder LC filter we get a band-stop filter by applying the same recipe as going from low-pass to band-pass.

These rules arise from requiring components to have the required impedance at important points of the frequency response: The centre frequency and the band edge. (Remember that a LPF is a BPF centred at f=0!)

Filter design from Ladder prototypes: component scaling

To scale the filter so it works at the required impedance level Z_0 ohms:

$$C' = C / Z_0 \;,\; L' = Z_0 L$$

To scale a low pass so that its break frequency is the required f_0 Hz:

$$C' f_0 = C \;,\; L' f_0 = L$$

After these transformations we can use the transformations from low pass to the required filter function as described before

Note:

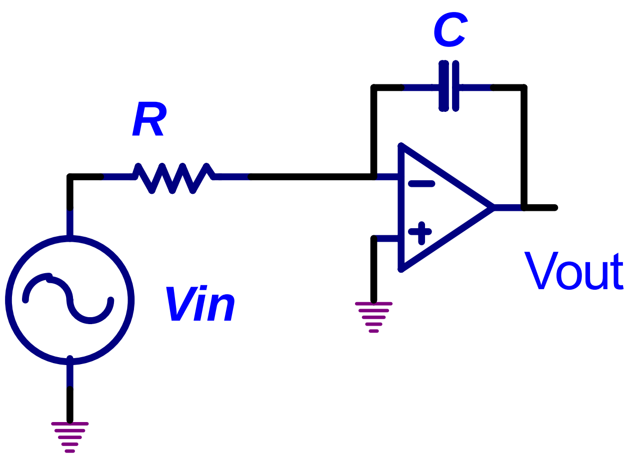
it is unusual to treat signal sources as pure voltage or current sources in professional engineering applications. (This would make circuits too noisy!)

In professional audio the standard impedance used is 600 Ohms.

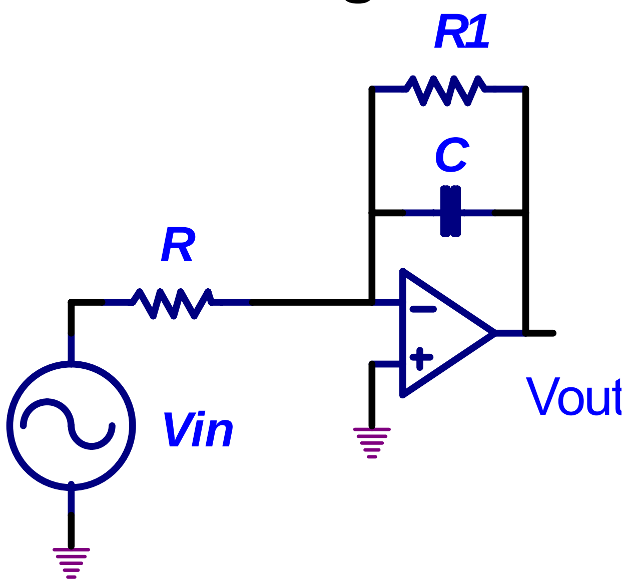
In cable, video and television applications the standard is 75 ohms

In most other radio frequency applications the standard is 50 ohms.

1st order low pass filter: the "Integrator"



"ideal" integrator



Lossy integrator

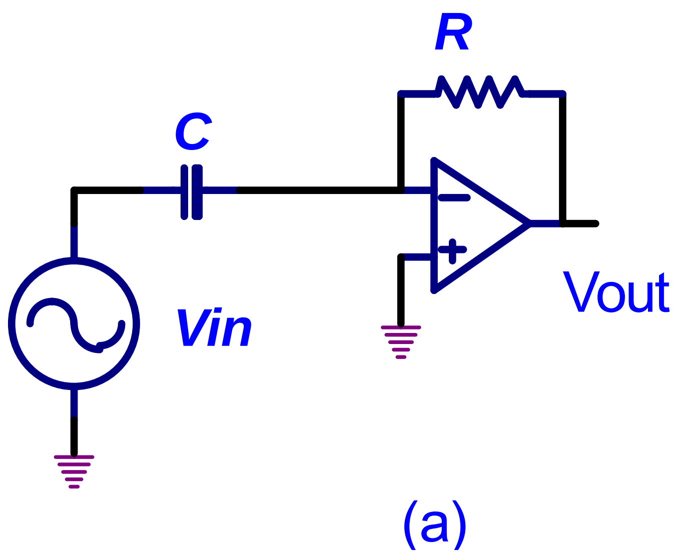
With ideal op-amp:

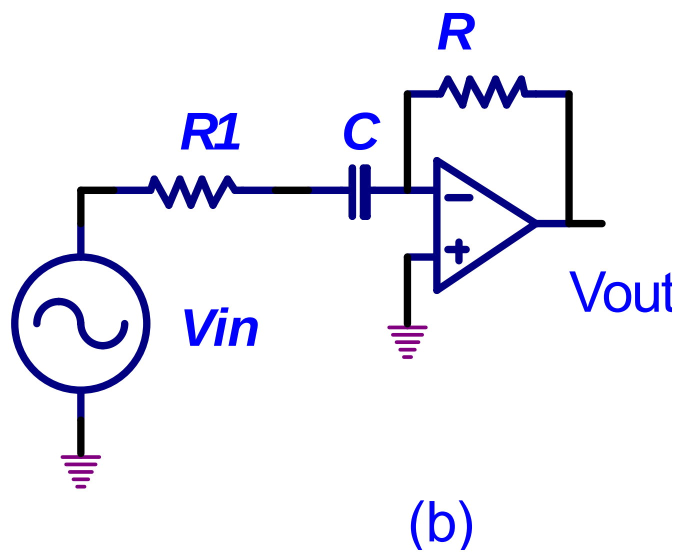
$$A_v = \frac{-1}{RC}$$

$$A_v = \frac{-R_1}{R}\frac{1}{1+j\omega R_1 C}$$

Note: *The ideal integrator is unstable at DC, and can only be used inside a feedback loop*

1st order high pass filter: the “differentiator”

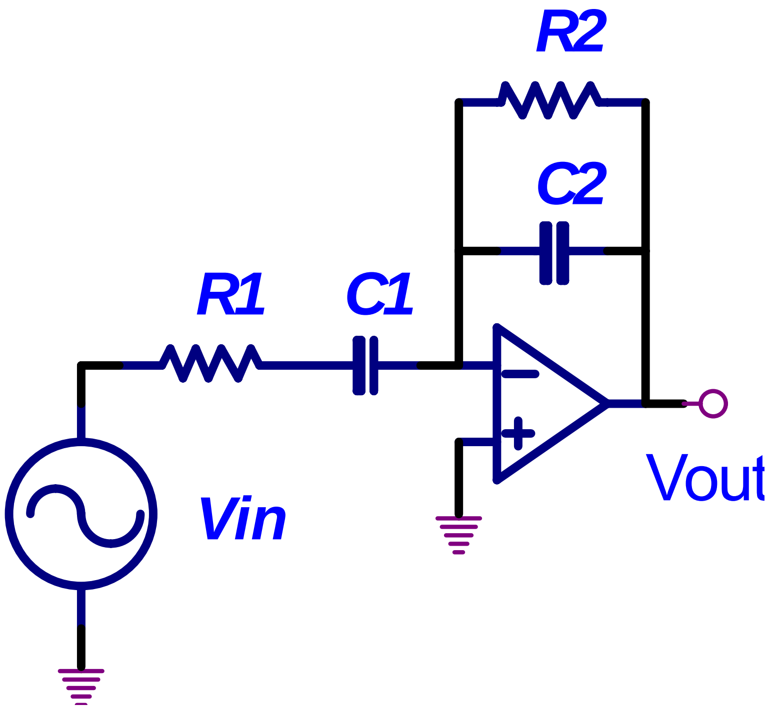




Ideal differentiator ***Lossy differentiator***

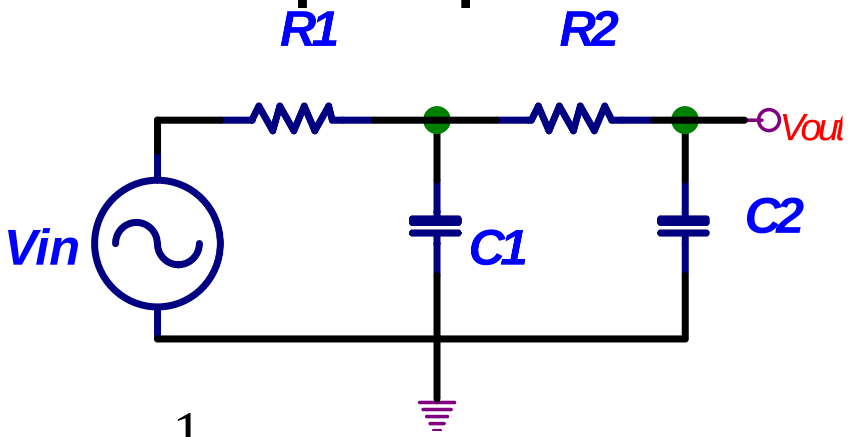
Note: The ideal differentiator when implemented with real op-amps becomes a very sharp Band Pass filter (lab, homework exercise)!

A simple band pass filter



Band pass filters are often a cascade of an LPF and an HPF,
In this example the op-amp acts both as a differentiator and an integrator.

2^{nd} order low pass passive RC filter



$$H(s) = \frac{1}{s^2 R_1 C_1 R_2 C_2 + s(R_1 C_1 + R_2 C_2 + R_1 C_2) + 1} = \frac{1}{s^2 \tau_1 \tau_2 + s(\tau_1 + \tau_2 + \tau_{12}) + 1}$$

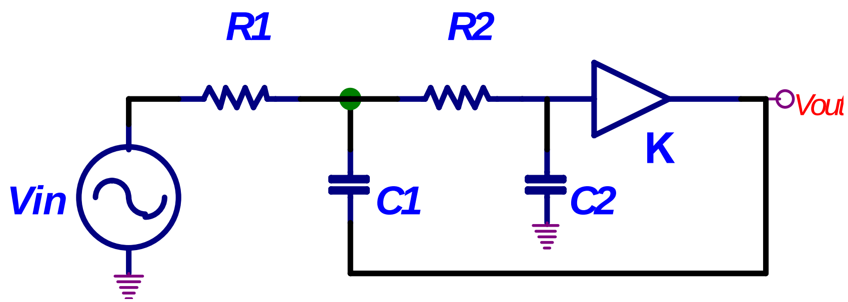
$$\omega_0 = 1/\sqrt{\tau_1 \tau_2} \quad , 2\zeta = \frac{1}{Q} = \sqrt{\frac{\tau_1}{\tau_2}} + \sqrt{\frac{\tau_2}{\tau_1}} + \sqrt{\frac{\tau_{12}}{\tau_{21}}} > 2 \Rightarrow Q < \frac{1}{2}$$

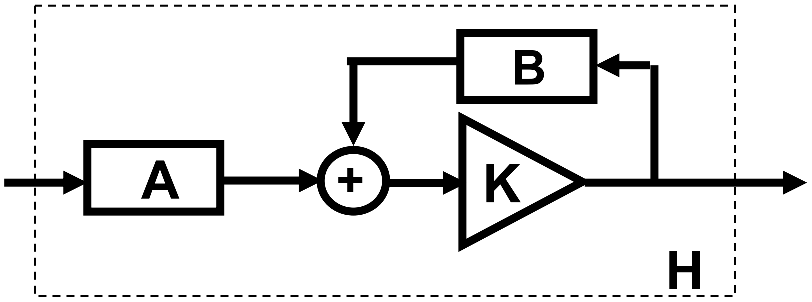
- **Since the minimum value of x+1/x is 2**
- **It follows that passive RC 2^{nd} order filters are OVERDAMPED**
- **The passive band pass filter transfer function calculation is part of experiment “Y” in the lab.**
- **Easiest way to analyse ladder networks is to construct successive Thevenin equivalent circuits starting from the source.**

Active RC Filters ("KRC")

- **The Q of a passive filter can be increased by the addition of feedback. In the following slides we will see several methods of doing this. The circuits are mostly known by the names of their inventors.**
- **Some common families of active filters are:**
  - ***The Sallen-Key filter (finite amplifier gain)***
  - ***The Deliyannis-Friend filters (assumes infinite amplifier gain)***
  - ***State variable filters, such as KHN (several amplifiers)***
  - ***Tow-Thomas Biquadratic filters (several amplifiers, several possible transfer functions, possible to electronically program the filter function)***
- **Note: Although we show these filters made with op-amps, they can be made with ANY amplifying device, e.g. with bipolar transistors or FETs.**
- **The actual device we use will have input and output impedance which we need to account for in the filter element value calculation.**

The Sallen Key Low Pass Filter (1)





By superposition, there are:

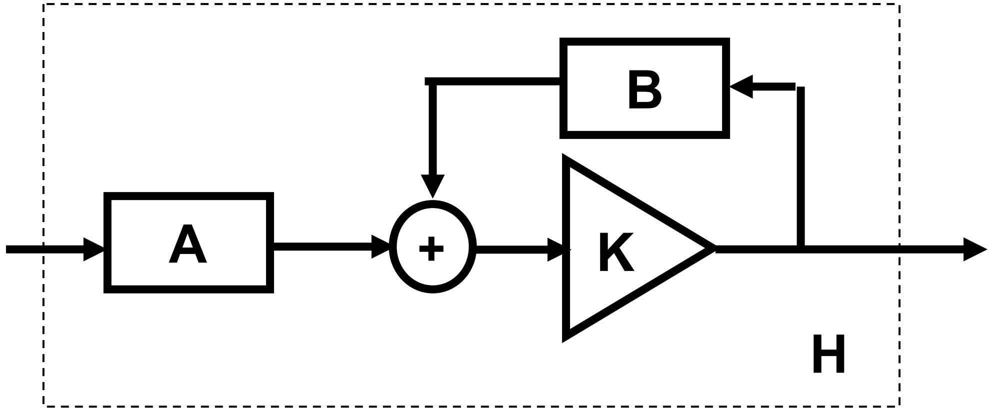
- An RC LPF in the forward signal path, of gain:

$$A = \frac{1}{s^2 R_1 C_1 R_2 C_2 + s\left(R_1 C_1 + R_1 C_2 + R_2 C_2\right) + 1}$$

- An RC BPF in the (positive) feedback path, reinforcing Q

$$B = \frac{sR_1 C_1}{s^2 R_1 C_1 R_2 C_2 + s\left(R_1 C_1 + R_1 C_2 + R_2 C_2\right) + 1}$$

The Sallen Key Low Pass filter (2)



From the block diagram it follows that

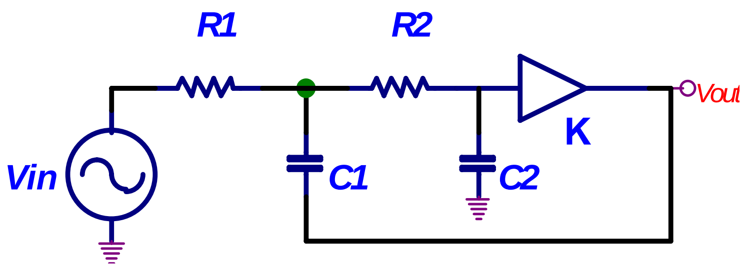
$$H = \frac{AK}{1 - BK}$$

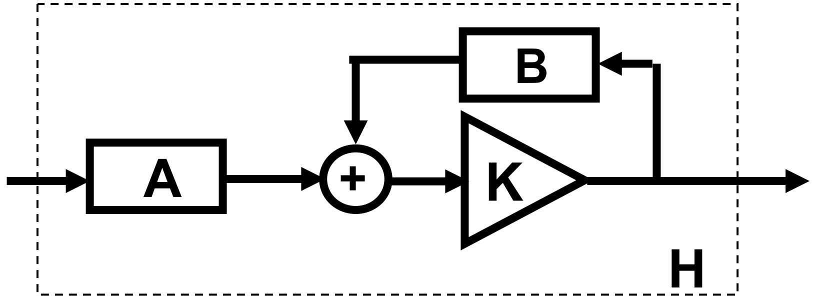
A and B are both rational functions, with the same denominator:

$$A = \frac{1}{Q(s)} \;,\; B = \frac{sR_1C_1}{Q(s)} \Rightarrow$$

$$H = \frac{K}{Q - KR_1C_1} = \frac{K}{s^2 R_1C_1R_2C_2 + s\left(\left(1-K\right)R_1C_1 + R_1C_2 + R_2C_2\right) + 1}$$

The Sallen Key Low Pass filter (3)





$$H = \frac{H_0}{s^2/\omega_n^2 + 2\zeta s/\omega_n + 1} = \frac{K}{s^2 R_1 C_1 R_2 C_2 + s\left(\left(1-K\right) R_1 C_1 + R_1 C_2 + R_2 C_2\right) + 1}$$

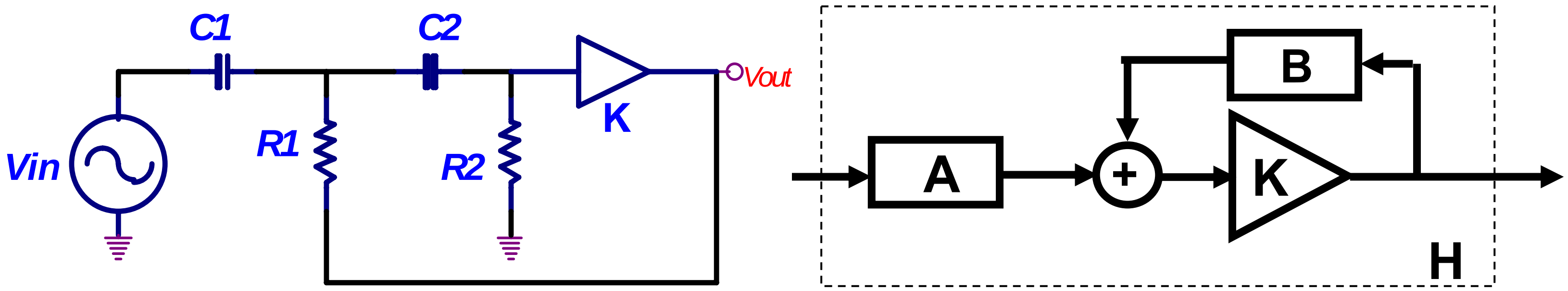
$$\frac{1}{\omega_n^2} = R_1 C_1 R_2 C_2 \Rightarrow \omega_n = \sqrt{\frac{1}{R_1 C_1 R_2 C_2}}$$

$$H_0 = K$$

$$\frac{2\zeta}{\omega_n} = \frac{1}{Q\omega_n} = \left(1-K\right) R_1 C_1 + R_1 C_2 + R_2 C_2 \Rightarrow 2\zeta = \frac{1}{Q} = \left(1-K\right)\sqrt{\frac{R_1 C_1}{R_2 C_2}} + \sqrt{\frac{R_2 C_2}{R_1 C_1}} + \sqrt{\frac{R_1 C_2}{R_2 C_1}}$$

For large enough K the circuit will have Q<0 and will become dynamically unstable, i.e. it will become an oscillator

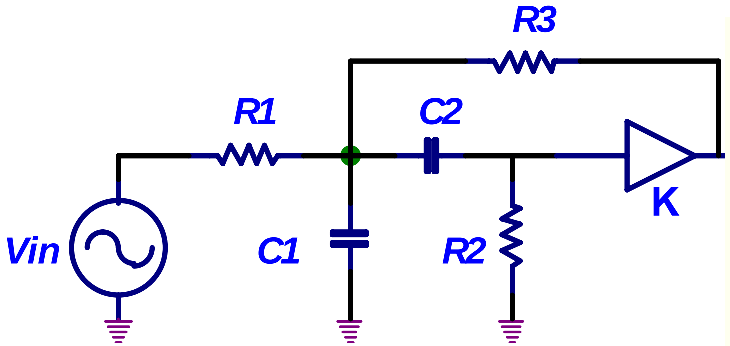
The Sallen Key High pass filter

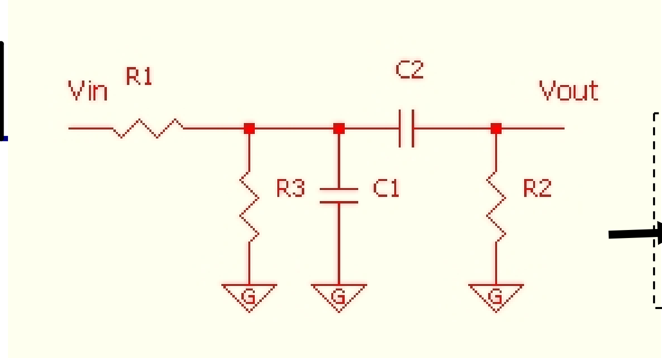


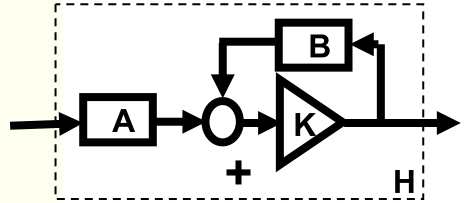
By superposition, there are:

- An RC HPF in the forward signal path
- An RC BPF in the (positive) feedback path, reinforcing Q
- Analysis very similar to that of the SK-LPF
- Detailed calculation left as a homework problem

The Sallen Key Band pass filter







This has identical in form passive band pass filters in the forward and feedback paths, shown on the middle. The block diagram in the right is the same form as the other SK filters. If $R_1=R_3$ then the two filters are identical and A=B . The transfer function of each path filter is:

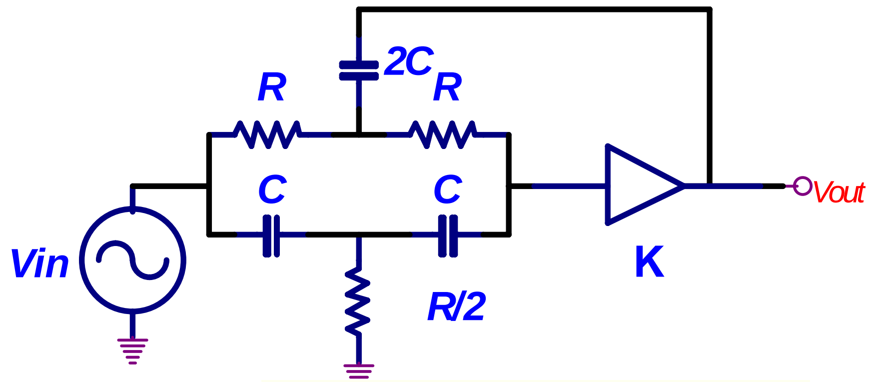
$$A = B = \frac{s\tau_2}{s^2\tau_1\tau_2 + \left(2\tau_2 + \tau_1 + \tau_{12}\right)s + 2} \ , \ \tau_1 = R_1C_1, \tau_2 = R_2C_2, \tau_{12} = R_1C_2$$

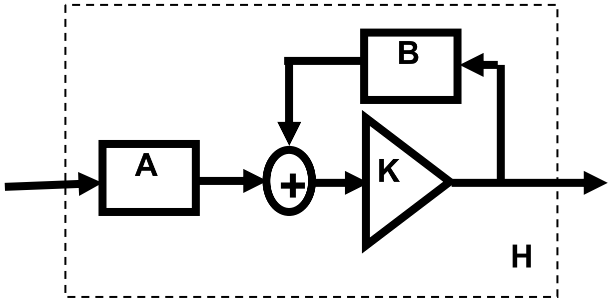
The entire SK filter has a transfer function:

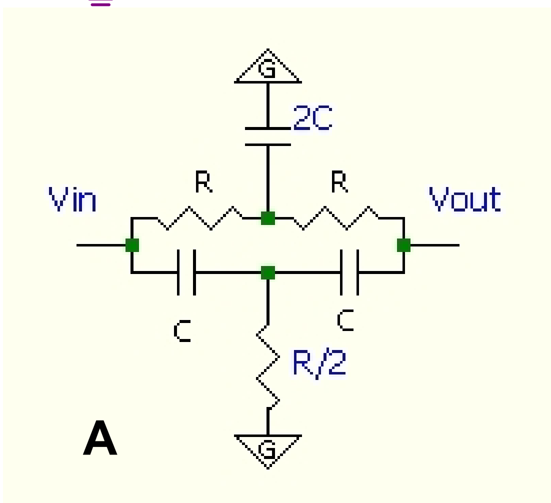
$$H = \frac{AH}{1-AH} = \frac{Ks\tau_2/2}{s^2\tau_1\tau_2/2 + \left(\left(2-K\right)\tau_2 + \tau_{12} + \tau_1\right)s/2 + 1}$$

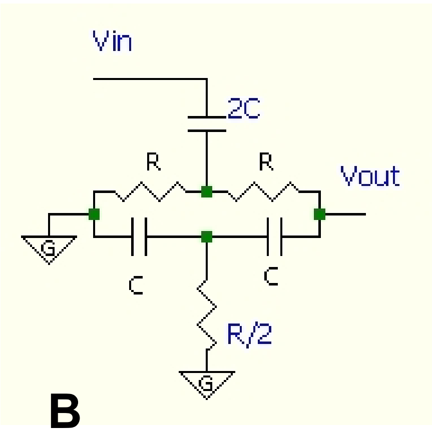
This circuit is studied in exercise 4 of the lab experiment ”Y”.

The Sallen Key Notch filter



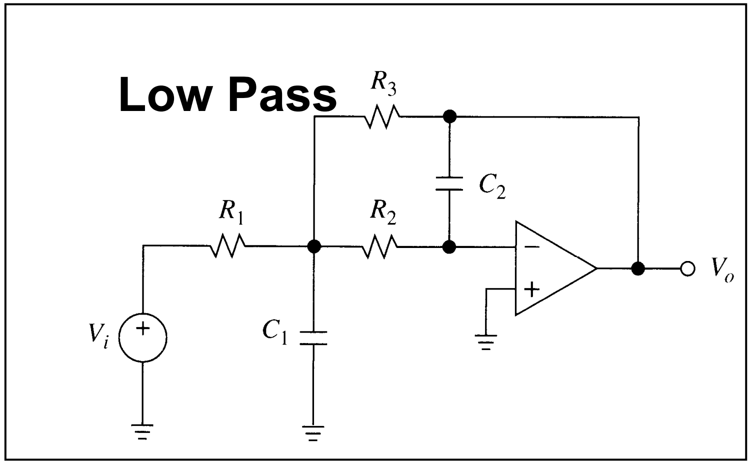


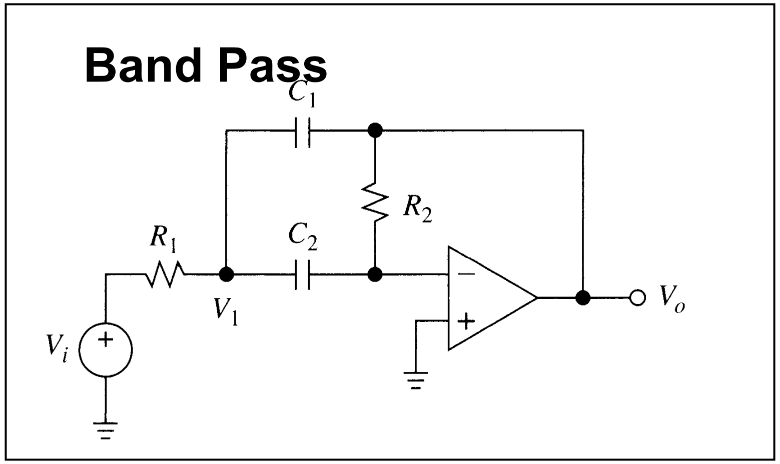


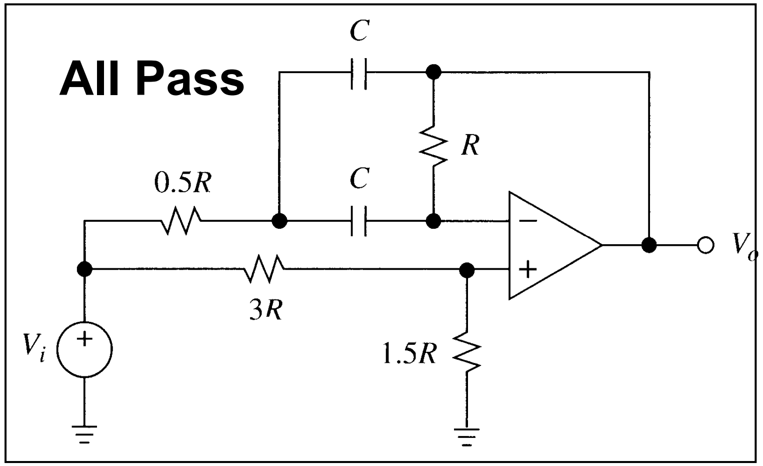


Networks A, B may be solved by nodal analysis or any other suitable method.

Multiple feedback filters: “Deliyannis-Friend” (“DF”)

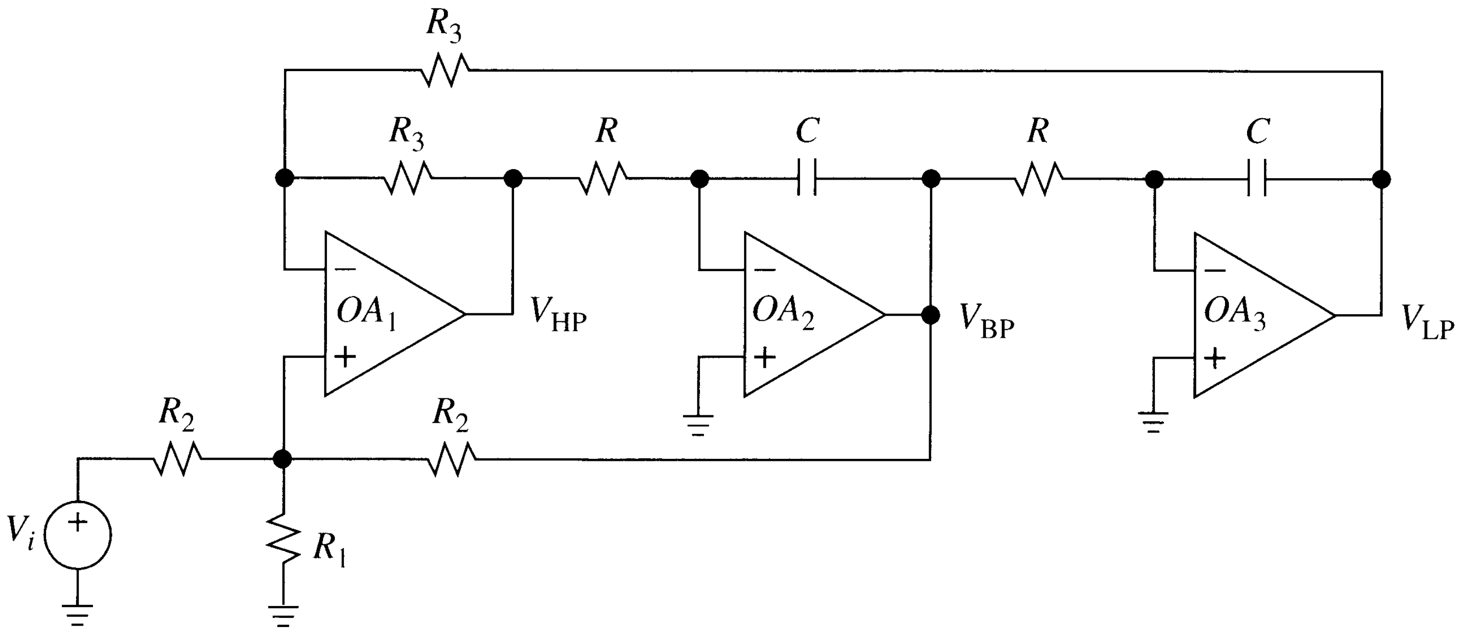






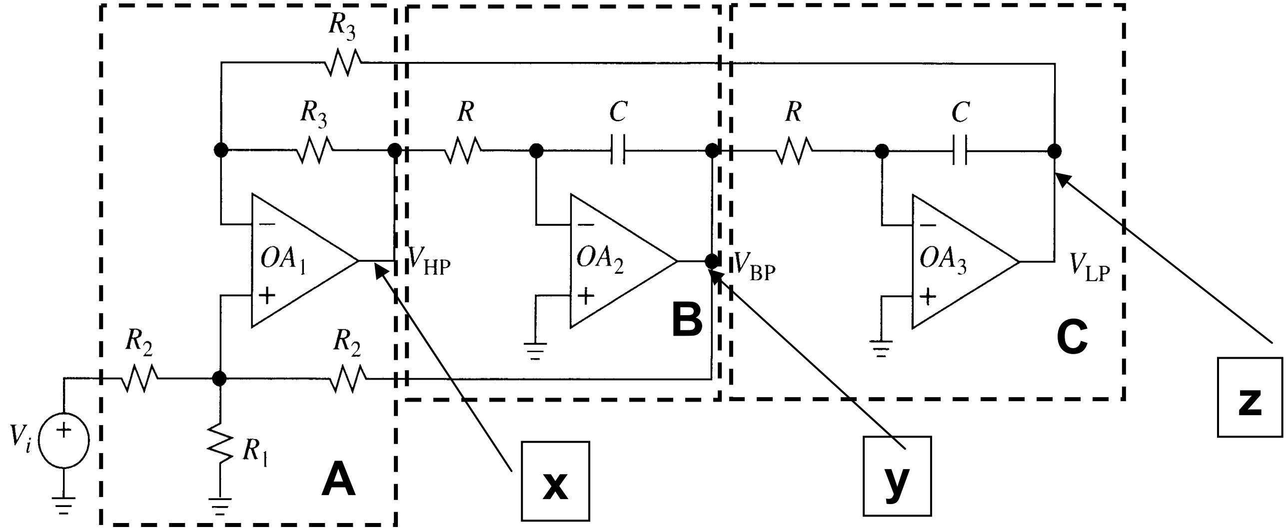
Op-amp is ideal→
Inverting input is virtual GND, V=0, i=0
Nodal analysis usually simple
Tee-Pi transforms may simplify algebra

“State Variable” filters - KHN



- “state variable filters” treat both the signal and its derivatives as variables
- A low pass filter performs time integration on signal waveforms
- A high pass filter performs time differentiation on signal waveforms
- Recall that filters are analogue computers which solve ODEs

"State Variable" filters – KHN : analysis



- **Block A is a weighted sum amplifier**
- **Blocks B and C are integrators**
- **Some maths: (after we get the constants K_1 , K_2, K_3 by nodal analysis)**

$$\tau = RC \ , K_1 = K_2 = \frac{2R_1 // R_2}{R_2 + R_1 // R_2}, K_3 = -1$$

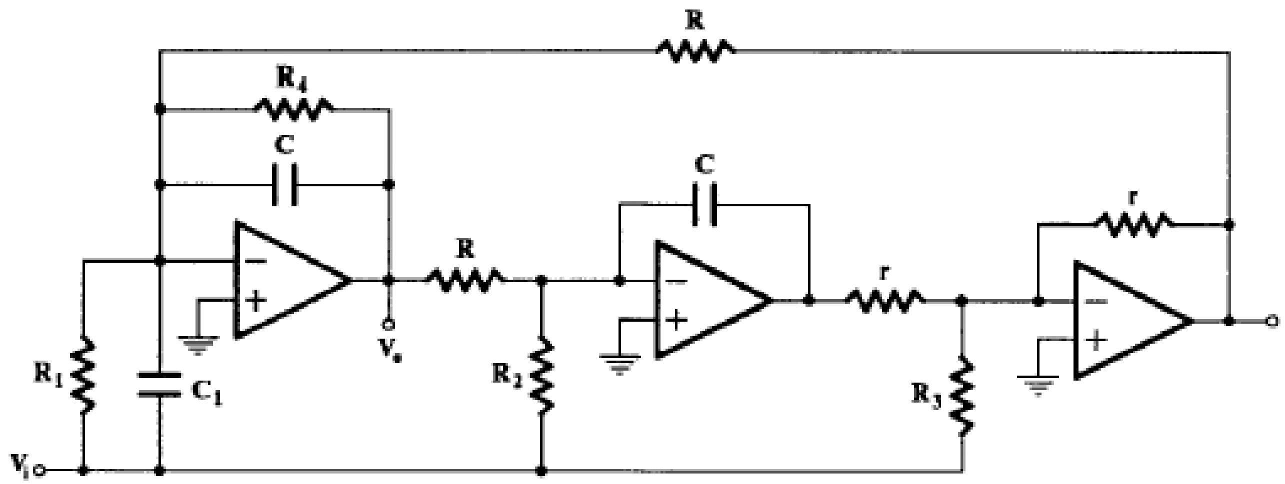
$$x = K_1 v_i + K_2 y - K_3 z \ , x = -\tau \dot{y} = \tau^2 \ddot{z} \Rightarrow$$

$$\tau^2 \ddot{z} - K_2 \tau \dot{z} + K_3 z = K_1 v_i \text{ (low pass filter)}$$

$$y = -\tau \dot{z} \text{ (Block C is an integrator, y is a BPF output)}$$

$$x = -\tau \dot{y} \text{ (Block B is an integrator, x is a HPF output)}$$

Another state variable filter: the Tow-Thomas "Biquad"



- the term "Biquadratic" or "Biquad" describes the 2nd order filter transfer function as a ratio of two quadratic polynomials
- R1, R2, R3 act as logical switches. Their presence or absence determines the filter function as Low, High or Band Pass →
- This is a single output universal filter; its function can be switched.
- The Tow Thomas filter an be treated:
  - By nodal analysis (easiest) or
  - As a "state variable" filter (note the two integrators and the summing operators)

Higher order filter synthesis using 2nd order sections

- **A general filter transfer function is of the form:**

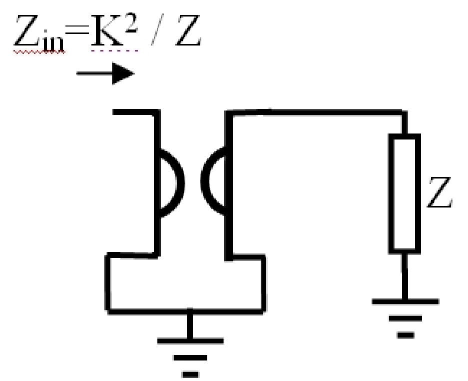
$$H(s) = \frac{P_n(s)}{Q_m(s)} = \frac{\sum_{i=0}^{n} a_k x^k}{\sum_{i=0}^{m} b_k x^k} = \frac{(s - z_0)(s - z_1)\cdots(s - z_n)}{(s - p_0)(s - p_1)\cdots(s - p_n)}$$

- **P(s) and Q(s) have real coefficients. To make a higher order filter:**
  - ***factor Q(s) into quadratic and linear factors***
  - ***Implement factors as biquads***
  - ***Cascade biquad sections to obtain the original transfer function***
  - ***Note that P and Q have real coefficients, so that their roots are either real or come in conjugate pairs.***
- **The centre frequencies and damping factors of the sections required to implement standard forms (Butterworth, Chebyshev, Elliptic etc) are tabulated in reference books.**
- **Tables are also included in CAD software for automated filter synthesis**

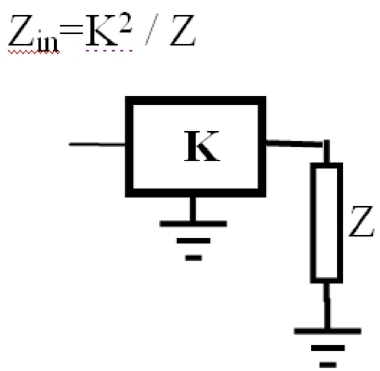
A useful network transformation: Impedance inversion and the gyrator

A gyrator can perform

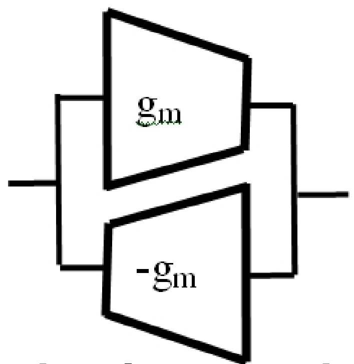
- impedance inversion (L←→C)
- Impedance scaling
- series ←→ parallel connection conversion!



"Proper" symbol of gyrator



Alternate symbol



Simple active implementation (very popular by analogue CMOS designers. Each gm is made of a MOSFET or two!)

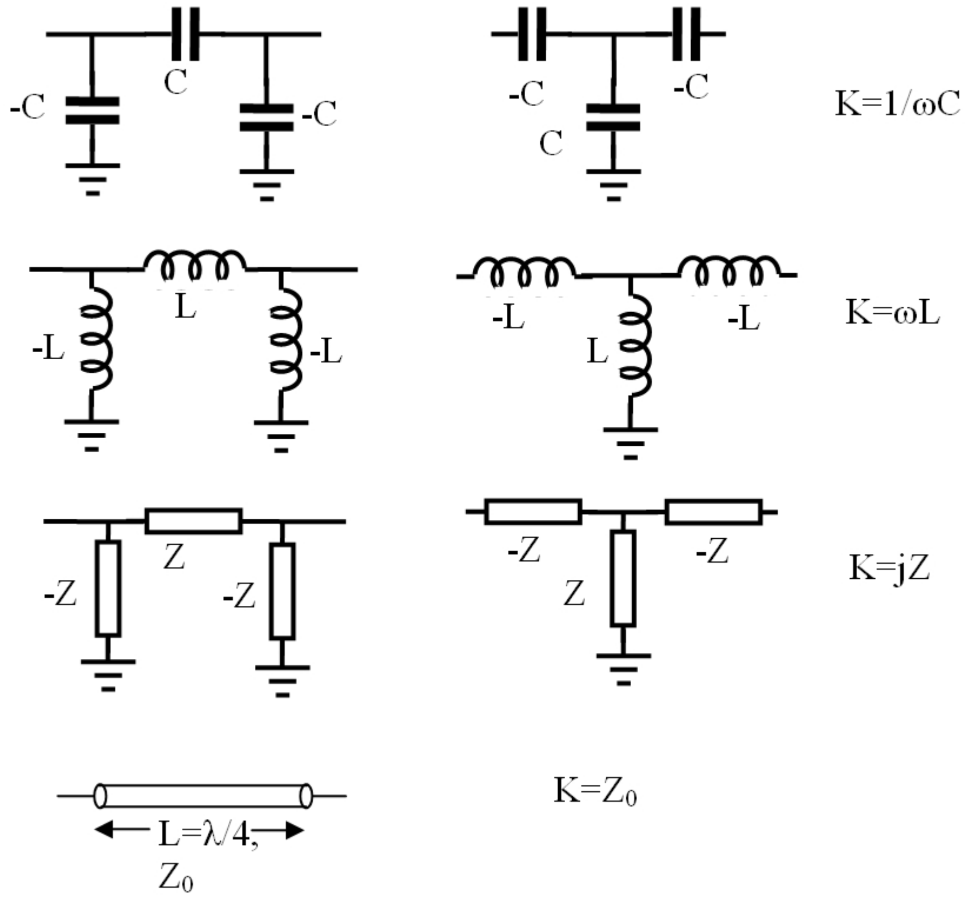
Passive Gyrators

- ¼ wavelength transmission line
- Pi and Tee networks with negative elements

negative values of components will be added to preceding and subsequent stage impedances resulting in overall positive impedances!

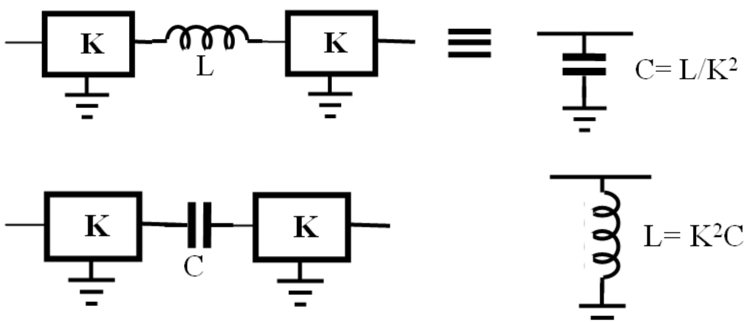
Ladder LC filters can be synthesised only with capacitors and gyrators

Z, -Z is completely arbitrary, can be a filter transfer function.

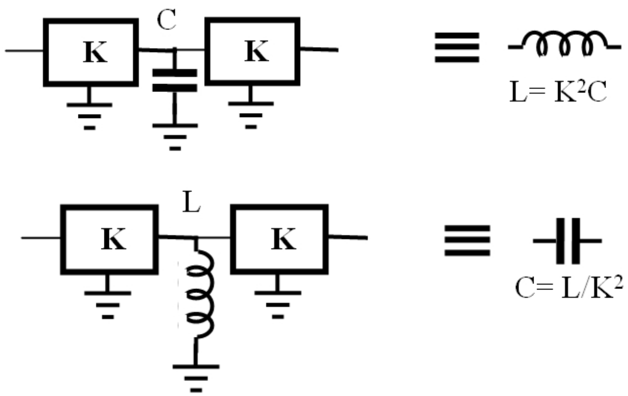


Gyrator function - basics

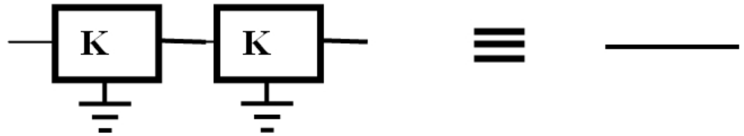
- A series (floating) component between two gyrators appears gyrated and grounded



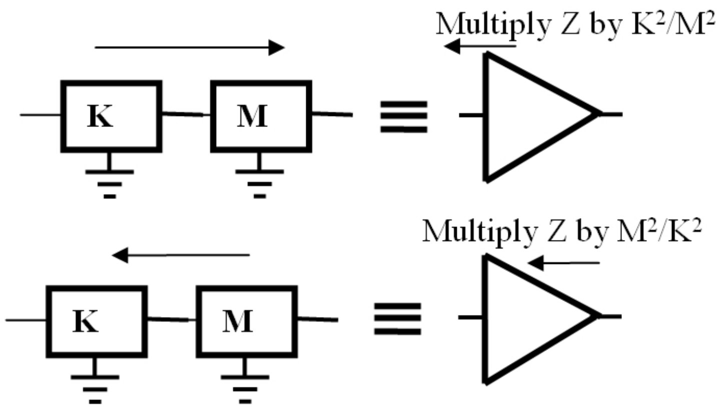
- A grounded component between two gyrators appears gyrated and in series



•Two identical gyrators in series are the identity operator

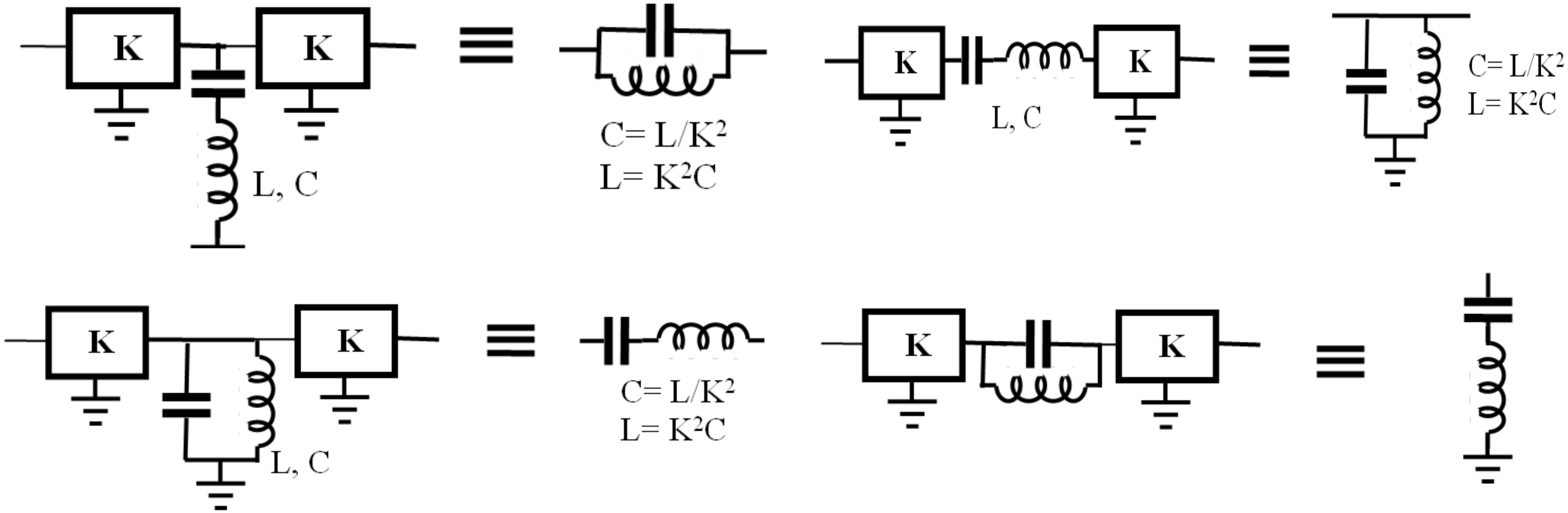


•Two different gyrators in series form a transformer, i.e. perform impedance scaling.

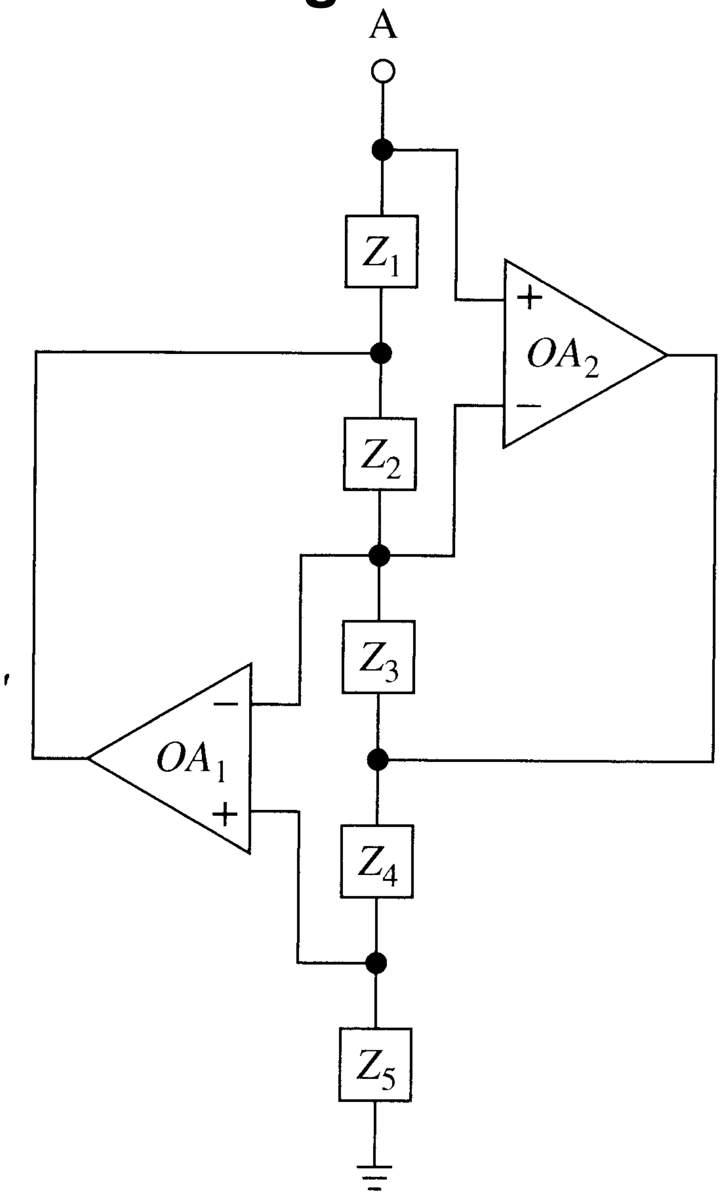


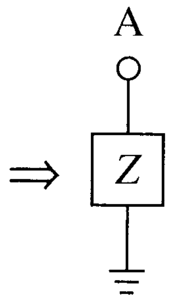
More gyrator identities

how to make e.g. a series resonance circuit when you only have parallel resonators in your component box… and vice versa



A generalised Impedance Converter ("GIC")





$$Z = \frac{Z_1 Z_3 Z_5}{Z_2 Z_4}$$

The GIC an be used as a gyrator to:

- **Synthesise L from C**
- **Synthesise C from L**
- **Synthesise a parallel LC from a series LC**
- **Synthesise a series LC from a parallel LC**
- **Scale component values**
- **Synthesise the FDNR (next slide)**

FDNR: the frequency dependent negative resistor

- **The filter transfer function of a circuit does not change if all components are multiplied by a constant K**
- **There is no requirement that the constant K is frequency independent!**
- **A useful multiplicative constant is**

$$K = 1/\, j\omega\tau$$

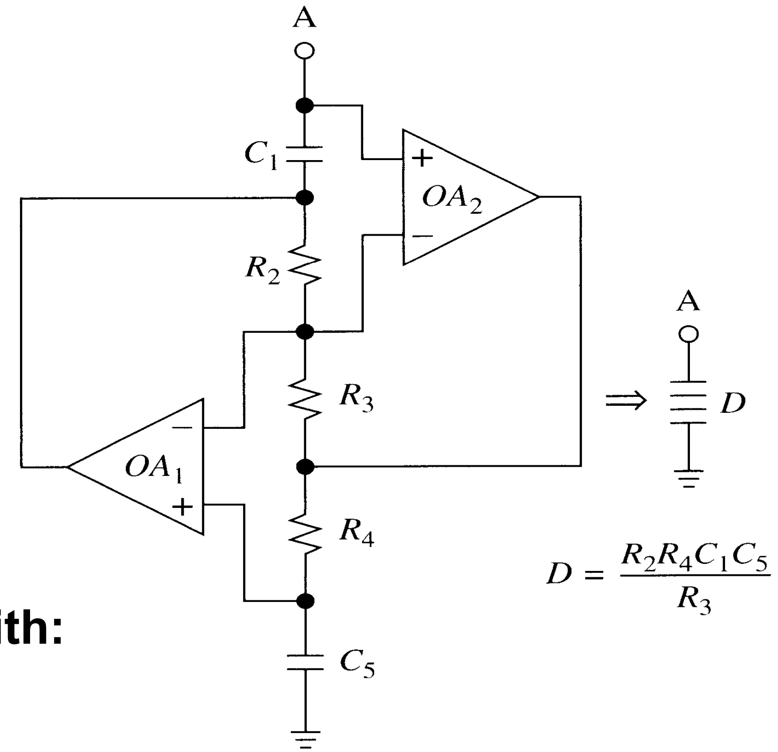
which

- **Transforms R → C**
- **Transforms L → R**
- **C → FDNR**

- **FDNR is a fictitious circuit element with:**

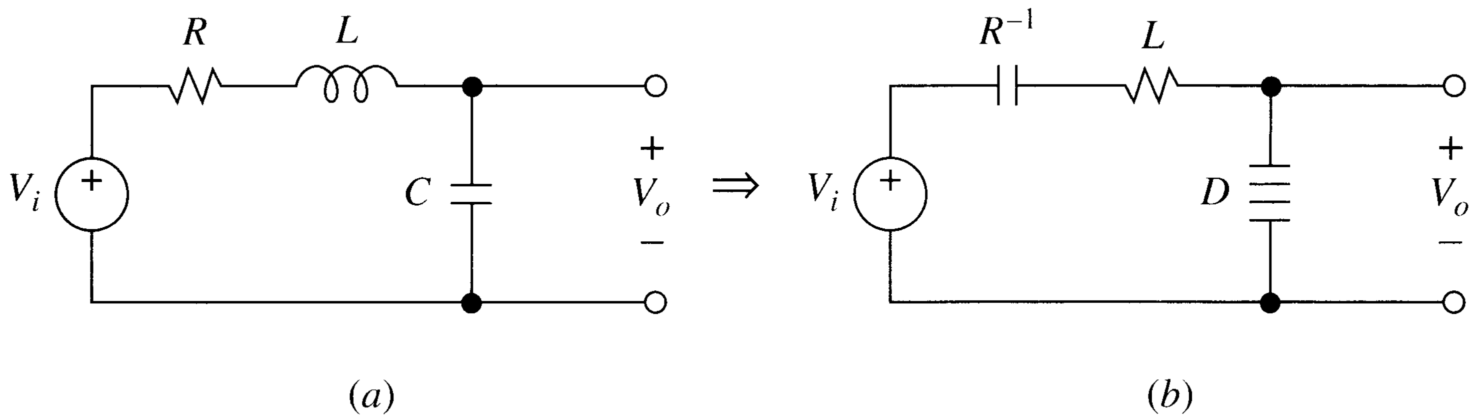
$$Y = -D\omega^2$$

- **A GIC can be used to implement an FDNR as illustrated on the right**
- **FDNR filters is one possible implementation of inductorless filters**



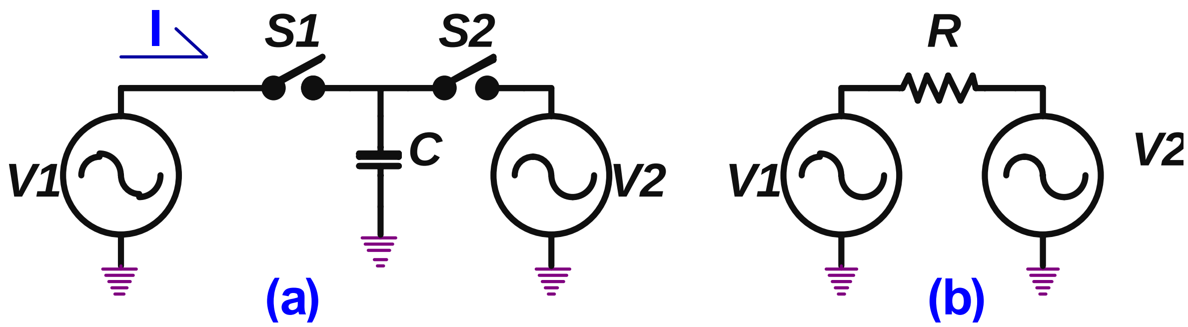
(b)

Example of FDNR transformation

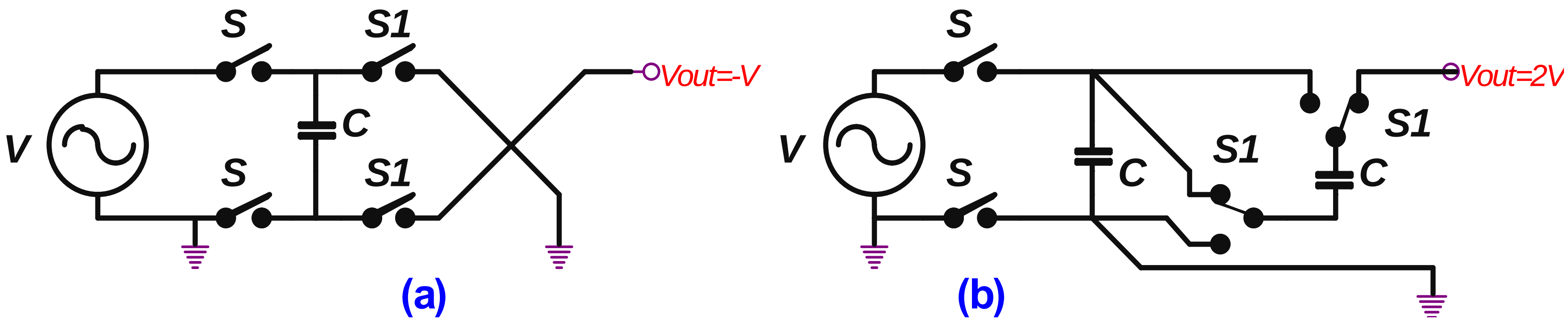


Note that we can scale the filter coefficients by any factor of our choice, including jω. All we need is that the voltage divider works as intended at all frequencies!

Switched Capacitor Filters: introduction

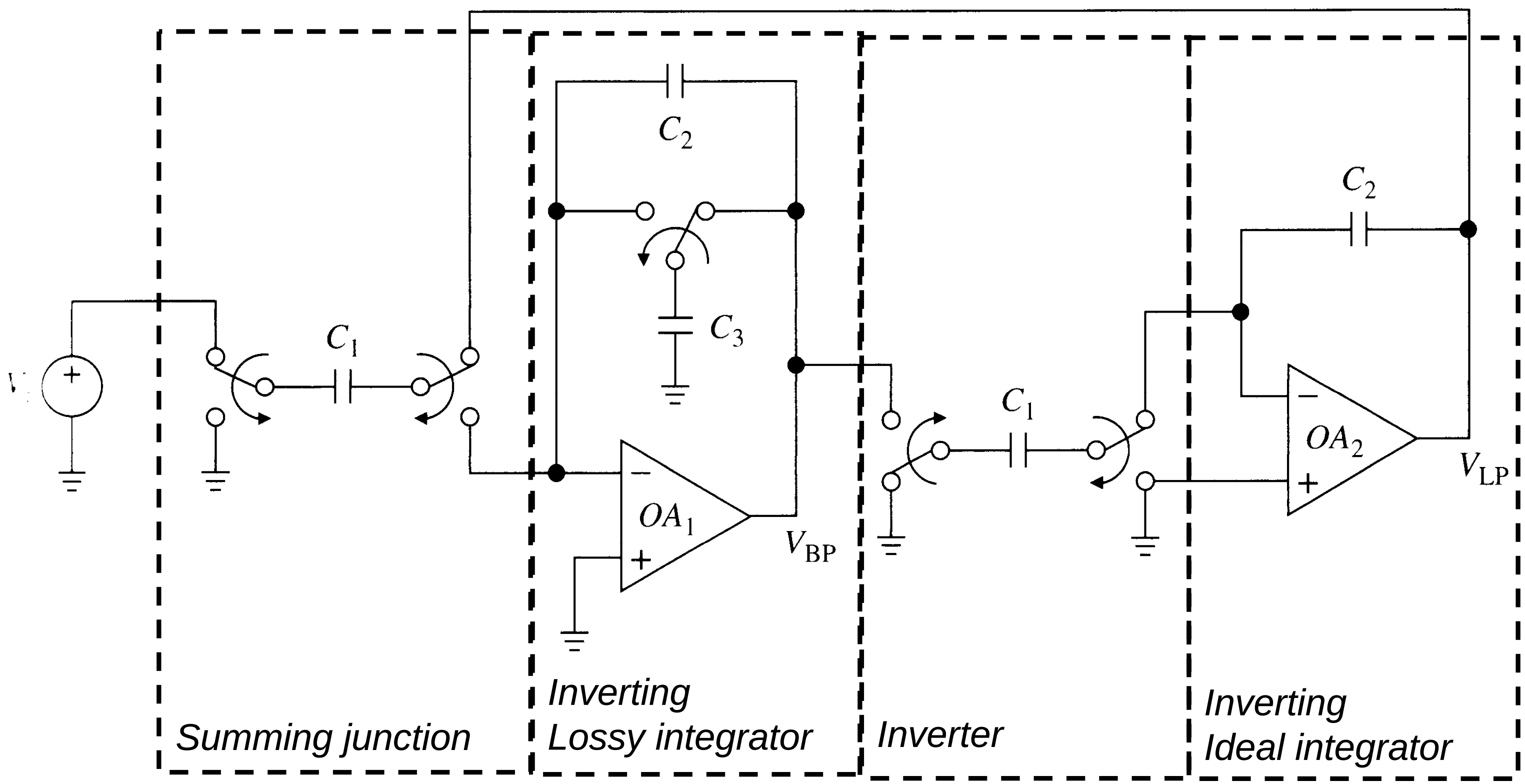


•(a) And (b) circuits are equivalent as long as signal frequency is much smaller than switching frequency
•The SC equivalent resistance is proportional to frequency



- Switched Cap circuits can be used for voltage amplification
- Switched Cap voltage amplifiers are called “charge pump” circuits
- examples of charge pump circuits: (a) V-gain=-1 , (b) V-gain=2

Switched Capacitor Biquads



- **Commercial chips contain several (typically 4) SC biquads in a package, which are then programmed and cascaded to synthesise higher order filters**
- **Frequencies of operation beyond audio (20kHz), typical constraint is product of f_o and Q. Switching frequenies in the MHz (need > 10x of highest f)**
- **This example has a structure similar to the Tow-Thomas**

Beyond KRLC: high Q filters

- Crystals. They behave in a circuit as series or parallel LC resonators:
  - *“Series mode” show an impedance minimum at resonance*
  - *“Parallel mode” show an impedance maximum at resonance*
  - *Quality factors very high*
  - *Low temperature variation, if necessary stabilised with “oven”*
- Dielectric Resonators
  - *A magnetic ceramic bead placed near a coil*
  - *Dimensions of bead determine frequency of resonance*
- Surface acoustic wave filters
  - *Printed conductor patterns on piezoelectric crystals*
  - *Filter function synthesised by interference of surface piezoelectric waves coupled to printed electrodes*
  - *Filter function extremely sensitive to source-load impedance*

Summary

- Types of filters: LP, HP, BP, BR, AP
- Transfer functions
- Bode Plots review
- Lumped element synthesis – Ladder filters
- Prototypes and transformations
- 1st order filters
- 2nd order filter transfer function
- Active filters: SK, DF, KHN, TT
- Gyrators and Generalised Impedance Converters
- Introduction to Switched capacitor filters