E2.2 Analogue electronics
Problem sheet 6 (Week 9)

Q1: Show that the impedance matrix of the "Tee" circuit (shown below) is: $\begin{bmatrix} Z_1 + Z_C & Z_C \\ Z_C & Z_2 + Z_C \end{bmatrix}$

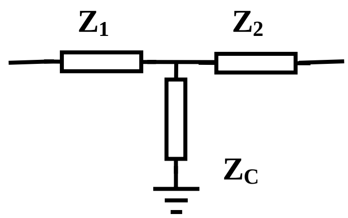

Figure for problem 3: The "tee" network

ANSWER:

$$z_{11} = \left.\frac{\partial v_1}{\partial i_1}\right|_{i_2=0} = Z_1 + Z_C \qquad z_{12} = \left.\frac{\partial v_1}{\partial i_2}\right|_{i_1=0} = Z_C$$

By definition, $z_{21} = \left.\frac{\partial v_2}{\partial i_1}\right|_{i_2=0} = Z_C \qquad z_{22} = \left.\frac{\partial v_2}{\partial i_2}\right|_{i_1=0} = Z_2 + Z_C$

Q2: Show that the impedance matrix of a network consisting of a 2-port with impedance matrix $Z_0 = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix}$ and a impedance Z_C along its common terminal connection is:

$$Z' = \begin{bmatrix} z_{11} + Z_C & z_{12} + Z_C \\ z_{21} + Z_C & z_{22} + Z_C \end{bmatrix}$$

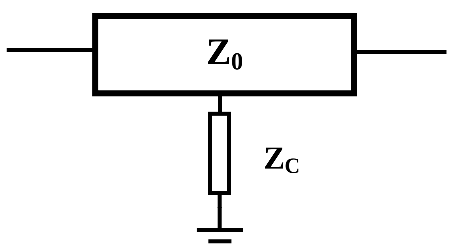

Figure for problem 4: Connecting an impedance along the common terminal of a 2-port.

ANSWER:

$$z_{11} = \left.\frac{\partial v_1}{\partial i_1}\right|_{i_2=0} = z_{11} + Z_C \qquad z_{12} = \left.\frac{\partial v_1}{\partial i_2}\right|_{i_1=0} = z_{12} + Z_C$$

By definition, $z_{21} = \left.\frac{\partial v_2}{\partial i_1}\right|_{i_2=0} = z_{21} + Z_C \qquad z_{22} = \left.\frac{\partial v_2}{\partial i_2}\right|_{i_1=0} = z_{22} + Z_C$

Q3: Prove that the "Tee" and "Pi" networks are equivalent if:

$$Z_1 = \frac{Y_2}{Y_1Y_2 + Y_2Y_3 + Y_1Y_3}\ ,\ Z_2 = \frac{Y_1}{Y_1Y_2 + Y_2Y_3 + Y_1Y_3}\ ,\ Z_3 = \frac{Y_3}{Y_1Y_2 + Y_2Y_3 + Y_1Y_3}$$ or, equivalently, if

$$Y_1 = \frac{Z_2}{Z_1Z_2 + Z_2Z_3 + Z_1Z_3}\ ,\ Y_2 = \frac{Z_1}{Z_1Z_2 + Z_2Z_3 + Z_1Z_3}\ ,\ Y_3 = \frac{Z_3}{Z_1Z_2 + Z_2Z_3 + Z_1Z_3}$$

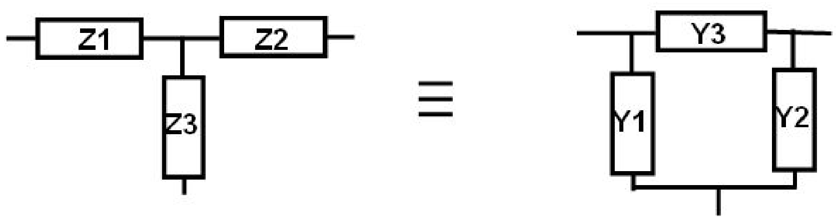

Figure for problem 5: Equivalence of Tee and Pi networks

ANSWER:

For the networks to be equivalent, it must be true that:

$$Y_{\text{Pi}} = \left(Z_{\text{Tee}}\right)^{-1} \Rightarrow$$

$$\begin{bmatrix} Y_1 + Y_3 & -Y_3 \\ -Y_3 & Y_2 + Y_3 \end{bmatrix} = \begin{bmatrix} Z_1 + Z_3 & Z_3 \\ Z_3 & Z_2 + Z_3 \end{bmatrix}^{-1} \Rightarrow$$

$$\begin{bmatrix} Y_1 + Y_3 & -Y_3 \\ -Y_3 & Y_2 + Y_3 \end{bmatrix} = \frac{1}{(Z_1 + Z_3)(Z_2 + Z_3) - Z_3^2}\begin{bmatrix} Z_2 + Z_3 & -Z_3 \\ -Z_3 & Z_1 + Z_3 \end{bmatrix} \Rightarrow$$

$$Y_1 = \frac{Z_2}{Z_1Z_2 + Z_2Z_3 + Z_1Z_3}\ ,\ Y_2 = \frac{Z_1}{Z_1Z_2 + Z_2Z_3 + Z_1Z_3}\ ,\ Y_3 = \frac{Z_3}{Z_1Z_2 + Z_2Z_3 + Z_1Z_3}$$

To prove the other set, we need to ask that $Z_{\text{Tee}} = \left(Y_{\text{Pi}}\right)^{-1}$

Q4: Given the Y-matrix for a network calculate its G-matrix

ANSWER

$$\begin{bmatrix} i_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}\begin{bmatrix} v_1 \\ i_2 \end{bmatrix} \Rightarrow \begin{bmatrix} y_{11} & y_{12} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ y_{21} & y_{22} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \Rightarrow$$

$$\begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} = \begin{bmatrix} y_{11} & y_{12} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ y_{21} & y_{22} \end{bmatrix}^{-1} = \frac{1}{y_{22}}\begin{bmatrix} y_{11} & y_{12} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} y_{22} & 0 \\ -y_{21} & 1 \end{bmatrix} \Rightarrow$$

$$\begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} = \frac{1}{y_{22}}\begin{bmatrix} \Delta y & y_{12} \\ -y_{21} & 1 \end{bmatrix}\ ,\ \Delta y = y_{11}y_{22} - y_{21}y_{12}$$

Q5: Given the G-matrix for a network calculate its Y-matrix

ANSWER

$$\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \Rightarrow \begin{bmatrix} g_{11} & g_{12} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} v_1 \\ i_2 \end{bmatrix} \Rightarrow$$

$$\begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} = \begin{bmatrix} g_{11} & g_{12} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ g_{21} & g_{22} \end{bmatrix}^{-1} = \frac{1}{g_{22}} \begin{bmatrix} g_{11} & g_{12} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} g_{22} & 0 \\ -g_{21} & 1 \end{bmatrix} \Rightarrow$$

$$\begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} = \frac{1}{g_{22}} \begin{bmatrix} \Delta g & g_{12} \\ -g_{21} & 1 \end{bmatrix} \text{ , } \Delta g = g_{11}g_{22} - g_{21}g_{12}$$

Q6: Use the results of Q4 and Q5 above to discover the exact form of the parallel form of the Miller theorem (what was shown in class was approximate!). What are the conditions for the approximation to be good?

ANSWER

The input admittance, when the output is open circuited comes from

$$i_2 = 0 \Rightarrow y_{21}v_1 + y_{22}v_2 = 0 \Rightarrow \frac{v_2}{v_1} = g_{21} = \frac{-y_{21}}{y_{22}}$$

so the output-open input admittance of any network is:

$$Y_{in}\Big|_{i_2=0} = \frac{i_1}{v_1} = y_{11} + y_{12}\frac{v_2}{v_1} = y_{11} + y_{12}g_{21}$$

Now we can calculate the input admittance of a network with a miller admittance y_c connected between its input and output. The Y matrix with the Miller Feedback element is:

$$Y' = \begin{bmatrix} y_{11} + y_c & y_{12} - y_c \\ y_{21} - y_c & y_{22} + y_c \end{bmatrix}$$

so the input admittance with the miller feedback element in place is:

$$Y_{in}\Big|_{i_2=0} = y_{11}' + y_{12}'g_{21}' = y_{11} + y_c + \left(y_{12} - y_c\right)g_{21}' =$$

$$= y_{11} + \left(1 - g_{21}'\right)y_c + y_{12}g_{21}'$$

The approximation is good if the amplifier is unilateral and the gain does not change when we connect the miller feedback element.