Adjoint networks and other elements of circuit theory

One-port reciprocal networks

- A one-port network is reciprocal if: $V^A I^B - I^A V^B = 0$

Where “A” and “B” are two different “tests” on the element

Example: a **linear** impedance (R,L,C) is reciprocal, since

$$\left.\begin{aligned} V^A &= ZI^A \\ V^B &= ZI^B \end{aligned}\right\} \Rightarrow V^A I^B - I^A V^B = ZI^A I^B - ZI^A I^B = 0$$

Note: A **non-linear** element will usually be non-reciprocal, since

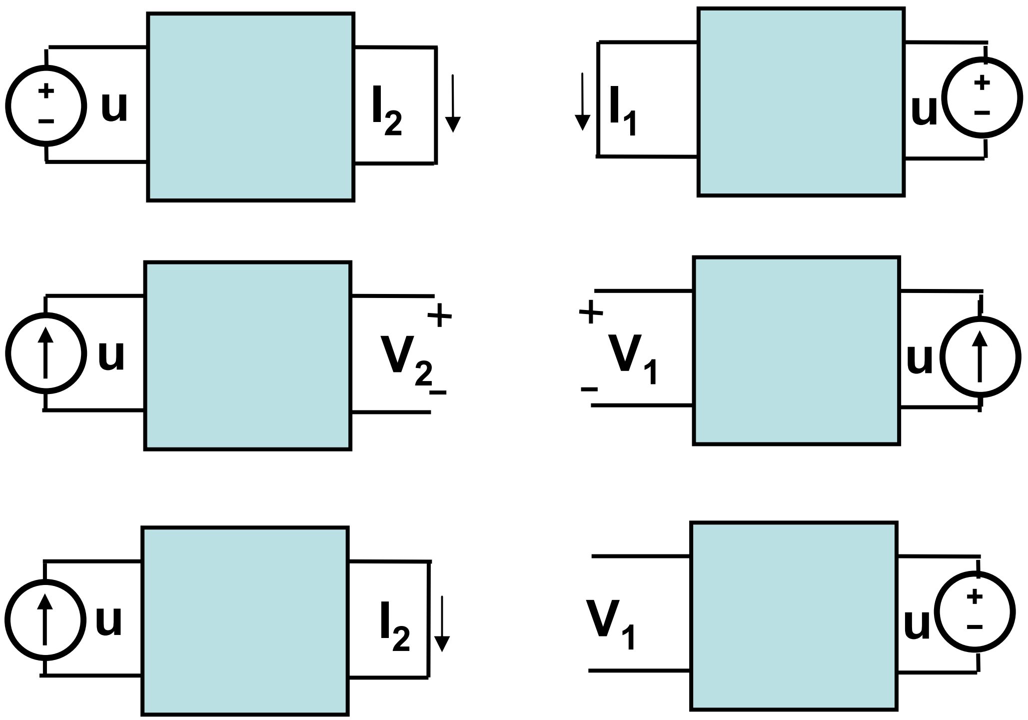
$$V = f\left(I\right)$$

And there is every reason to expect that at two different operating points:

$$V^A I^B - I^A V^B = f\left(I^A\right) I^B - f\left(I^B\right) I^A \neq 0$$

2-port reciprocal networks

3 statements:



$I_1 = I_2$

$V_1 = V_2$

$V_1 = I_2$

n-port reciprocal networks

•Can be proven that an n-port network is reciprocal if:

$$\sum_n \left(V_n^A I_n^B - I_n^A V_n^B \right) = 0$$

The summation taken over the n-ports

If the network is linear we can write the vector I in terms of V and the admittance matrix:

$$\sum_n V_n^A I_n^B - I_n^A V_n^B = \begin{bmatrix} V_1^A & \cdots & V_n^A \end{bmatrix} \overleftrightarrow{\mathbf{Y}} \begin{bmatrix} V_1^B \\ \\ V_n^B \end{bmatrix} - \begin{bmatrix} V_1^A & \cdots & V_n^A \end{bmatrix} \overleftrightarrow{\mathbf{Y}}^T \begin{bmatrix} V_1^B \\ \\ V_n^B \end{bmatrix}$$

We conclude that a n-port is reciprocal if it is linear and possesses a symmetric admittance matrix.

More on reciprocal networks

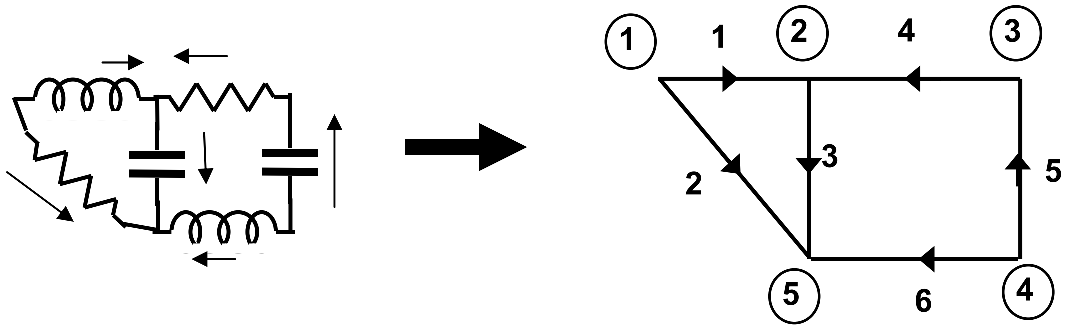
- Networks made out of resistors, capacitors, inductors, and mutual inductors are reciprocal
- Networks which include controlled sources and gyrators are not reciprocal
- We focus on two-port networks. Reciprocal networks have:

$$y_{21} = y_{12} \ , z_{21} = z_{12} \ , h_{21} = -h_{12} \ , g_{21} = -g_{12} \ ,$$

$$AD - BC = 1$$

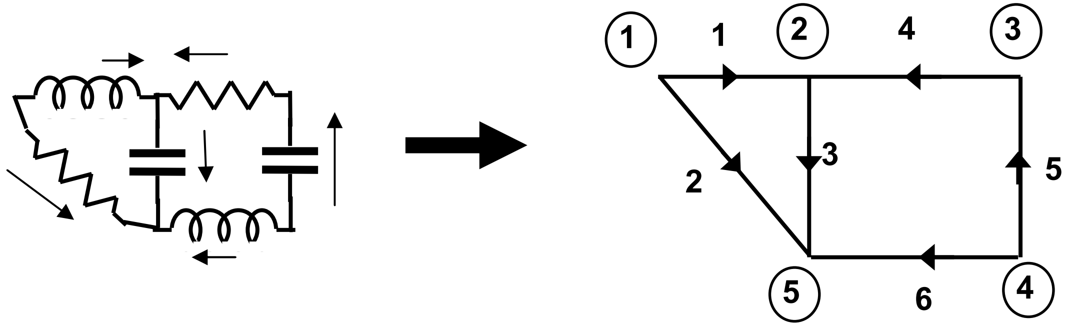
Graphs and incidence matrices

- Abstract a circuit as a directed graph
  - Each circuit element becomes a directed line



  - Write the incidence matrix:
    - # nodes as rows
    - # branches as columns
  - Entries:
    - 1 if branch leaves node, -1 if branch arrives at node, 0 if no connection

The incidence matrix



- The incidence matrix of the above is

$$
\text{5 nodes}\left\{\begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix}\right.
\overset{\overbrace{\begin{matrix}1 & 2 & 3 & 4 & 5 & 6\end{matrix}}^{\text{6 branches}}}{\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & -1 & -1 & 0 & 0 & -1 \end{bmatrix}} = \mathbf{A}
$$

KCL and Tellegen’s theorem

- Let I be the column vector of the branch currents
- **Kirchhoff's current law** is: (!)
  - $I^T A=[0]$ (row vector of zeroes, one for each node)
  - $AI=[0]$ (column vector of zeroes, one for each node)
- **Tellegen’s Theorem:**

If two networks A,B have the same incidence matrix, then:

$$\sum_{n \in \text{branches}} V_n^A I_n^B = 0$$

If A, B refer to the same network this is the statement of conservation of power: As much power as generated by sources will be dissipated by the rest of the network.

Exercise: show that any network with internal RLC components only is reciprocal. (hint: apply Tellegen’s theorem with an input applied only to P1 or to P2 and argue that the rest of the terms in the sum are identical for both configurations; after all it IS the same network, so it does have the same graph!)

Adjoint networks

- Two n-port networks A, B are "inter-reciprocal" or "adjoint" if:

$$\sum_n \left(V_n^A I_n^B - I_n^A V_n^B \right) = 0$$

- The previous observations apply: Two networks are adjoint if:

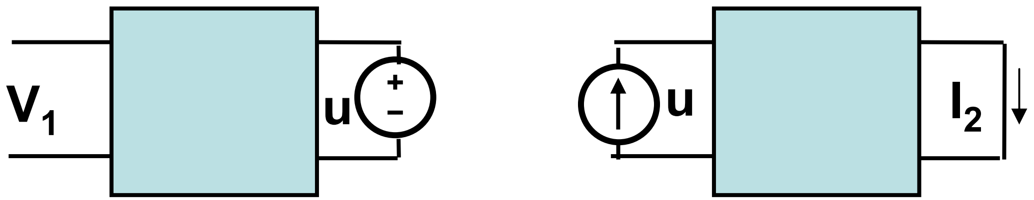
$$y_{21}^A = y_{12}^B \;,\; z_{21}^A = z_{12}^B \;,\; g_{21}^A = -g_{12}^B \;,\; h_{21}^A = -h_{12}^B$$

Note that the reverse gain of a voltage amplifier (g parameters) is a current gain
Also, the reverse gain of a current amplifier (h parameters) is a voltage gain

We expect, therefore the adjoint of a voltage amplifier to be a reversely connected voltage amplifier which acts like a current amplifier.

The adjoint of an ideal voltage amplifier

- From the reciprocity theorem it is evident that a current amplifier of the same gain will be the adjoint of a voltage amplifier.

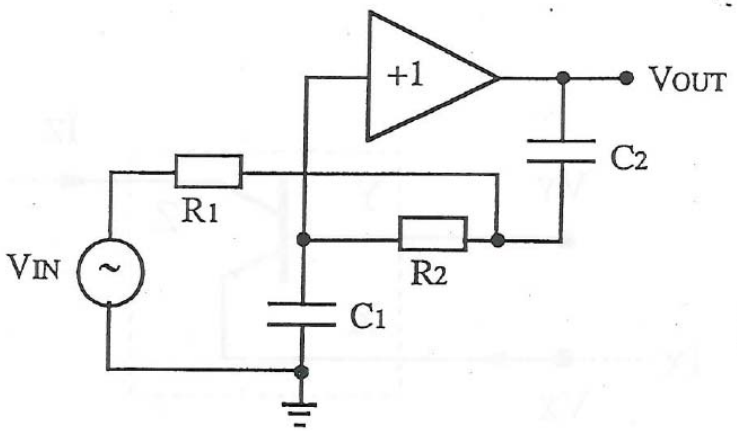


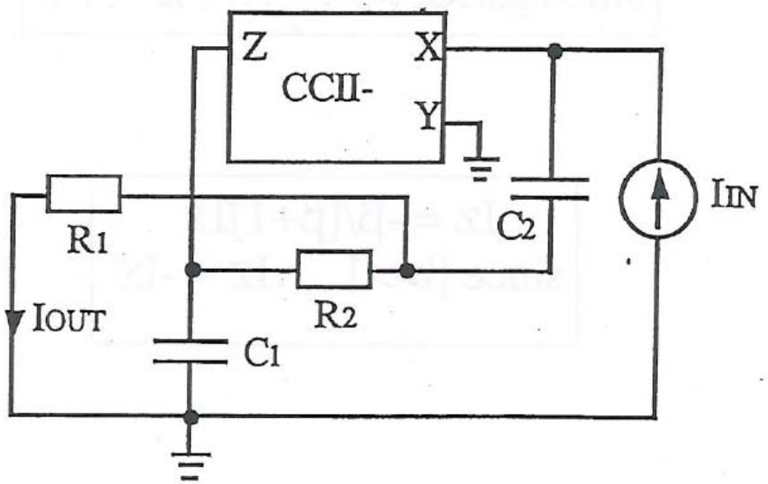
$V_1=I_2$

Note that the thevenin impedances of sources and loads remain the same!
Also note that the current gain is the **negative of the voltage gain**

Constructing the adjoint of a circuit

- Leave the resistors, capacitors, and inductors as they are.
- Replace voltage sources by short circuits (i.e. current sinks – the inputs become outputs) (and vice-versa)
- Replace voltage meters by current sources (i.e. the outputs become inputs (and vice-versa)
- The following 2 circuits have the same function, one in voltage, the other in current mode. Botha are Sallen-Kev filters





2nd order filter transfer functions ("Biquads"): Review

Second order filter transfer functions are all of the following form:

$$H(s) = H_0 \frac{C(s/\omega_0)^2 + 2B\zeta s/\omega_0 + A}{(s/\omega_0)^2 + 2\zeta s/\omega_0 + 1} \ , \ Q = \frac{1}{2\zeta}$$

H_0 is the overall amplitude, ω_0 the break (or peak) frequency, and ζ the damping factor

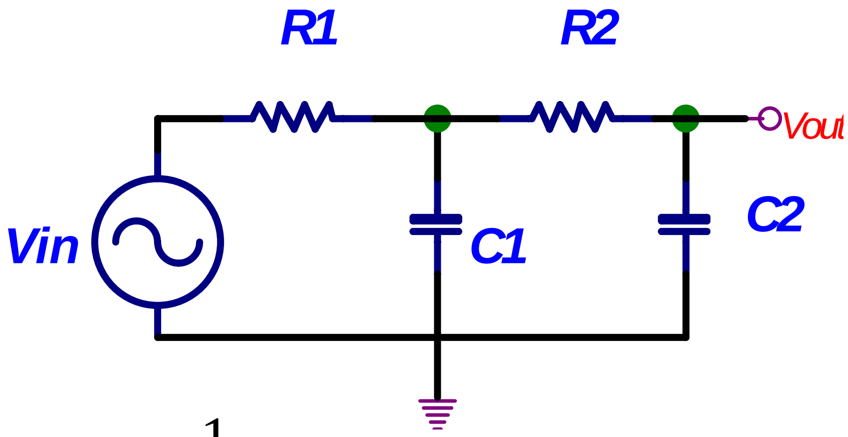
ζ isrelated to the quality factor Q by: $Q=1/2\zeta$

The 3dB bandwidth of an underdamped 2nd order filter is approx 1/Q times the peak frequency.

The coefficients A, B, C determine the function of the filter:

Function	A	B	C
Low Pass	1	0	0
High Pass	0	0	1
Band Pass	0	1	0
Band Stop	1	0	1
All Pass	1	-1	1

2^{nd} order low pass passive RC filter

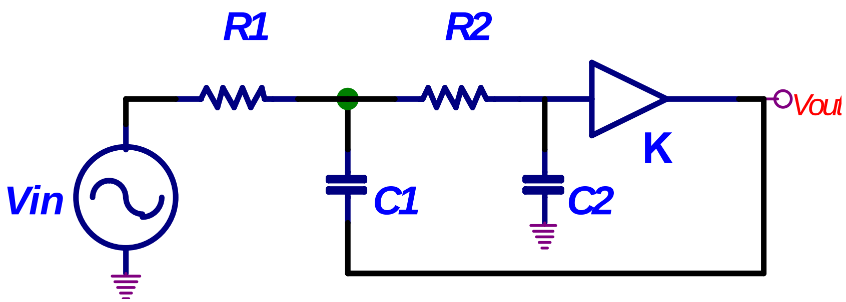


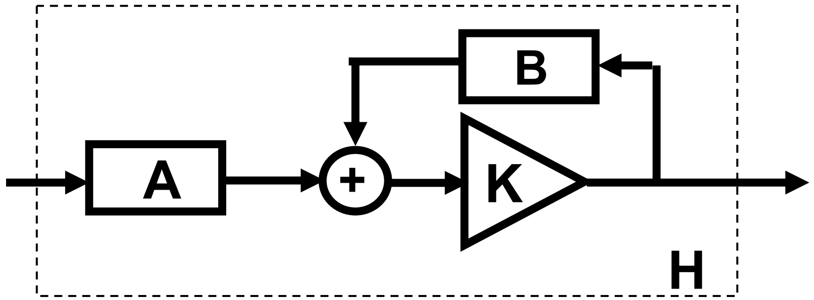
$$H(s)=\frac{1}{s^2R_1C_1R_2C_2+s\left(R_1C_1+R_2C_2+R_1C_2\right)+1}=\frac{1}{s^2\tau_1\tau_2+s\left(\tau_1+\tau_2+\tau_{12}\right)+1}$$

$$\omega_0=1/\sqrt{\tau_1\tau_2}\quad,2\zeta=\frac{1}{Q}=\sqrt{\frac{\tau_1}{\tau_2}}+\sqrt{\frac{\tau_2}{\tau_1}}+\sqrt{\frac{\tau_{12}}{\tau_{21}}}>2$$

- **Since the minimum value of x+1/x is 2**
- **It follows that passive RC 2^{nd} order filters are OVERDAMPED**
- **The passive band pass filter transfer function calculation is part of experiment “Y” in the lab.**

The Sallen Key Low Pass Filter (1)





By superposition, there are:

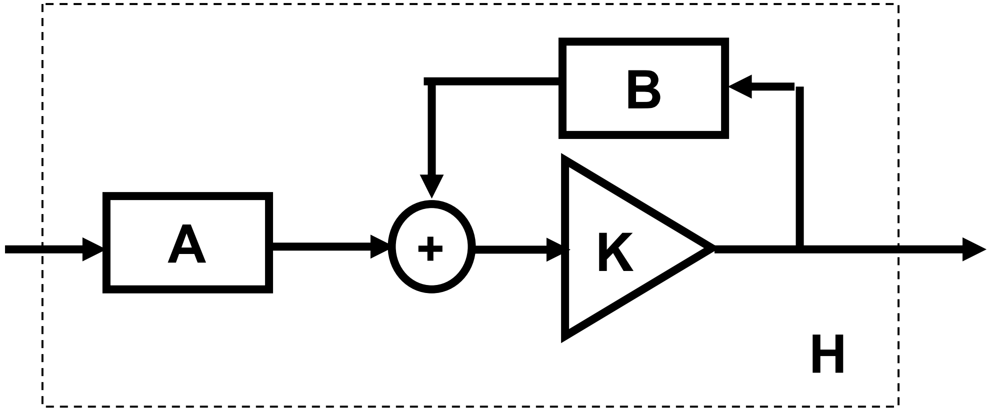
- An RC LPF in the forward signal path, of gain:

$$A = \frac{1}{s^2 R_1 C_1 R_2 C_2 + s\left(R_1 C_1 + R_1 C_2 + R_2 C_2\right) + 1}$$

- An RC BPF in the (positive) feedback path, reinforcing Q

$$B = \frac{s R_1 C_1}{s^2 R_1 C_1 R_2 C_2 + s\left(R_1 C_1 + R_1 C_2 + R_2 C_2\right) + 1}$$

The Sallen Key Low Pass filter (2)



From the block diagram it follows that

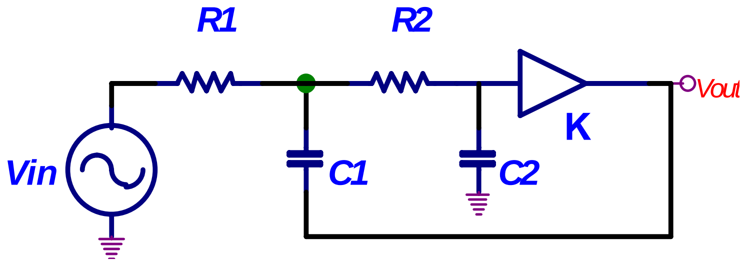
$$H = \frac{AK}{1 - BK}$$

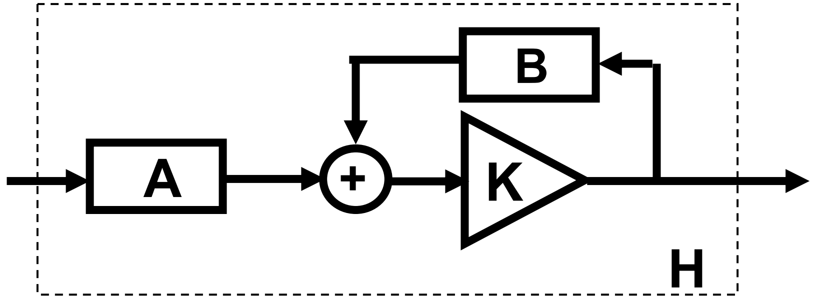
A and B are both rational functions, with the same denominator:

$$A = \frac{1}{Q(s)} \;, B = \frac{sR_1C_1}{Q(s)} \Rightarrow$$

$$H = \frac{K}{Q - KR_1C_1} = \frac{K}{s^2R_1C_1R_2C_2 + s\left(\left(1-K\right)R_1C_1 + R_1C_2 + R_2C_2\right) + 1}$$

The Sallen Key Low Pass filter (3)





$$H = \frac{H_0}{s^2/\omega_0^2 + 2\zeta s/\omega_0 + 1} = \frac{K}{s^2 R_1C_1R_2C_2 + s\left(\left(1-K\right)R_1C_1 + R_1C_2 + R_2C_2\right) + 1}$$

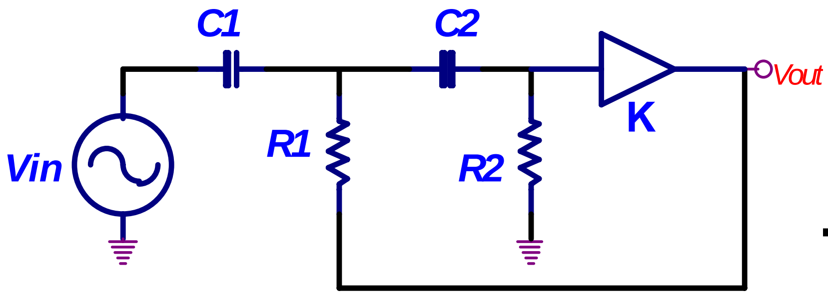
$$\frac{1}{\omega_0^2} = R_1C_1R_2C_2 \Rightarrow \omega_0 = \sqrt{\frac{1}{R_1C_1R_2C_2}}$$

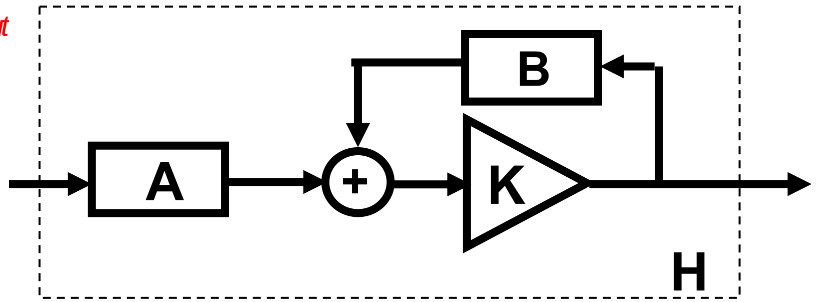
$$H_0 = K$$

$$\frac{2\zeta}{\omega_0} = \frac{1}{Q\omega_0} = \left(1-K\right)R_1C_1 + R_1C_2 + R_2C_2 \Rightarrow 2\zeta = \frac{1}{Q} = \left(1-K\right)\sqrt{\frac{R_1C_1}{R_2C_2}} + \sqrt{\frac{R_2C_2}{R_1C_1}} + \sqrt{\frac{R_1C_2}{R_2C_1}}$$

For large enough K the circuit will have Q<0 and will become dynamically unstable, i.e. it will become an oscillator

The Sallen Key High pass filter

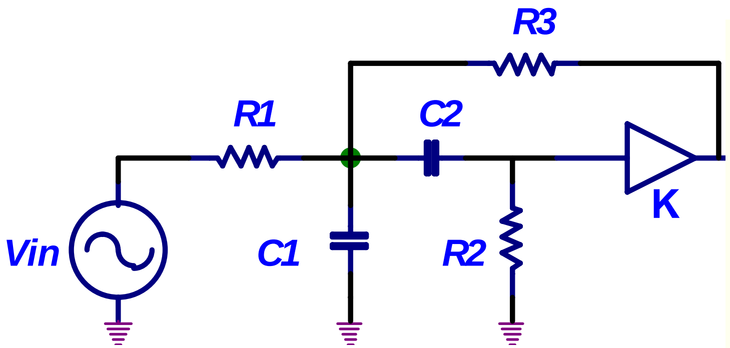


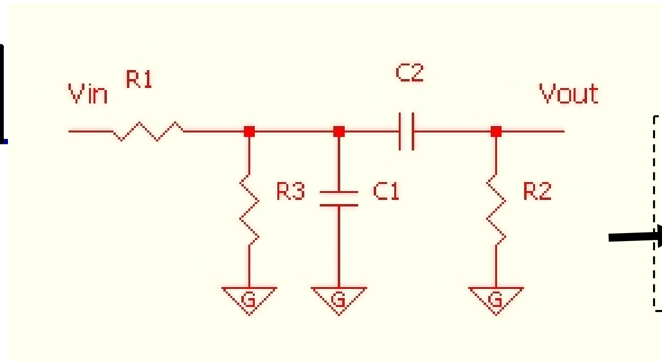


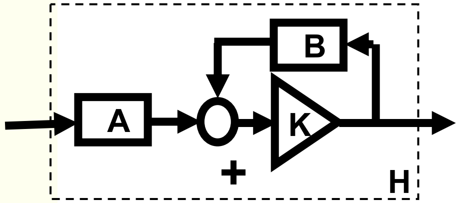
By superposition, there are:

- An RC HPF in the forward signal path
- An RC BPF in the (positive) feedback path, reinforcing Q
- Analysis very similar to that of the SK-LPF
- Detailed calculation left as a homework problem

The Sallen Key Band pass filter







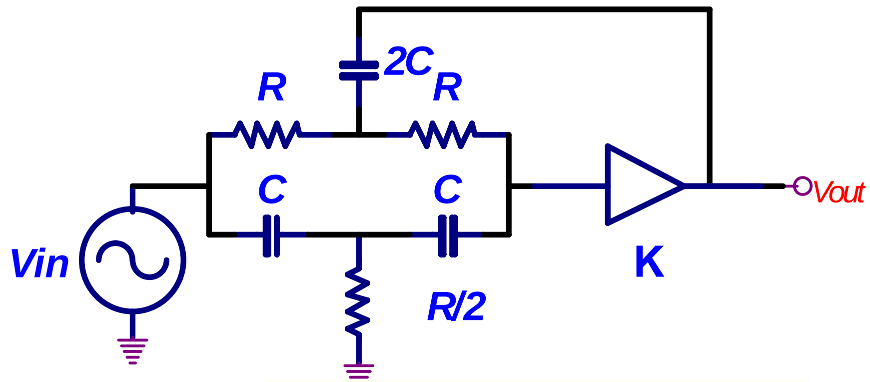
This has identical in form passive band pass filters in the forward and feedback paths, shown on the middle. The block diagram in the right is the same form as the other SK filters. If R_1=R_3 then the two filters are identical and A=B . The transfer function of each path filter is:

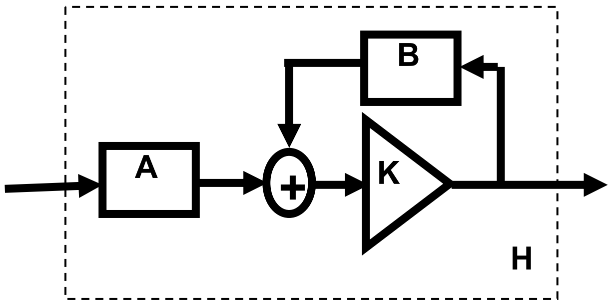
$$A = B = \frac{s\tau_2}{s^2\tau_1\tau_2 + \left(2\tau_2 + \tau_1 + \tau_{12}\right)s + 2}\ , \ \tau_1 = R_1C_1, \tau_2 = R_2C_2, \tau_{12} = R_1C_2$$

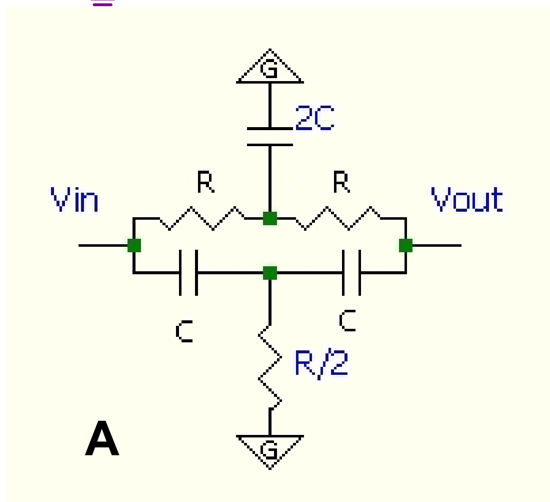
The entire SK filter has a transfer function:

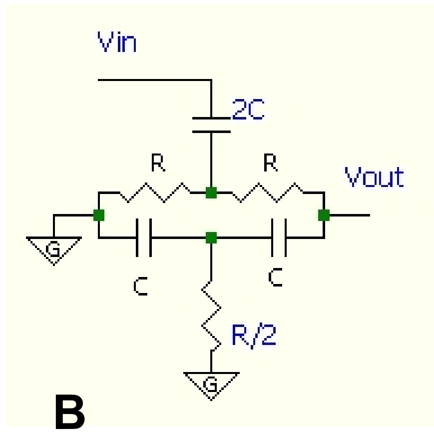
$$H = \frac{AH}{1 - AH} = \frac{Ks\tau_2/2}{s^2\tau_1\tau_2/2 + \left(\left(2-K\right)\tau_2 + \tau_{12} + \tau_1\right)s/2 + 1}$$

The Sallen Key Notch filter







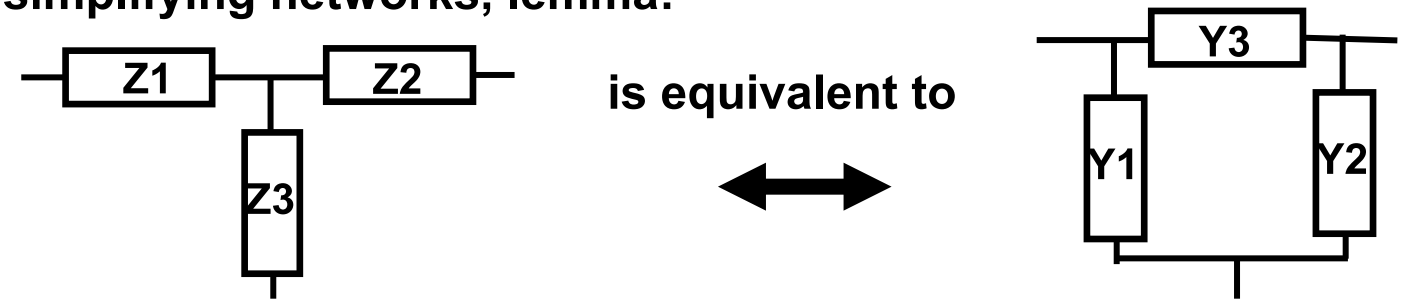


Networks A, B may be solved by nodal analysis or any other suitable method.

Tee – Pi transformations – “dual” networks

When analysing active band pass or band stop active filters we often encounter the “twin-tee” passive notch filter topology

This requires quite a bit of algebra to compute, so we prove a, useful for simplifying networks, lemma:



Proof: write the z matrix of the Tee< and the y matrix of the Pi and require that the two circuits are representations of same network:

$$\mathbf{Z}_{Tee} = \begin{bmatrix} Z_1 + Z_3 & Z_3 \\ Z_3 & Z_2 + Z_3 \end{bmatrix} \text{ , } \mathbf{Y}_{Pi} = \begin{bmatrix} Y_1 + Y_3 & -Y_3 \\ -Y_3 & Y_2 + Y_3 \end{bmatrix}, \mathbf{Y}_{Pi} = \mathbf{Z}_{Tee}^{-1} \Rightarrow$$

$$\Rightarrow \begin{cases} Z_1 = \dfrac{Y_2}{Y_1Y_2 + Y_1Y_3 + Y_2Y_3} \text{ , } Z_2 = \dfrac{Y_1}{Y_1Y_2 + Y_1Y_3 + Y_2Y_3} \text{ , } Z_3 = \dfrac{Y_3}{Y_1Y_2 + Y_1Y_3 + Y_2Y_3} \\ Y_1 = \dfrac{Z_2}{Z_1Z_2 + Z_1Z_3 + Z_2Z_3} \text{ , } Y_2 = \dfrac{Z_1}{Z_1Z_2 + Z_1Z_3 + Z_2Z_3} \text{ , } Y_3 = \dfrac{Z_3}{Z_1Z_2 + Z_1Z_3 + Z_2Z_3} \end{cases}$$