Large signal linearisation

Dynamic range is usually traded off for power

Usual solutions, reducing the Power-Dyn. range conflict:

- Feedback:
  - Series (emitter Degeneration)
  - Shunt (miller feedback)
- Interpolation
  - Connecting in parallel devices at different operating points
  - Connecting in parallel devices of different sizes
- Here we examine intrinsically large signal linear circuits
  - Dynamic translinear circuits **LOG DOMAIN filters**

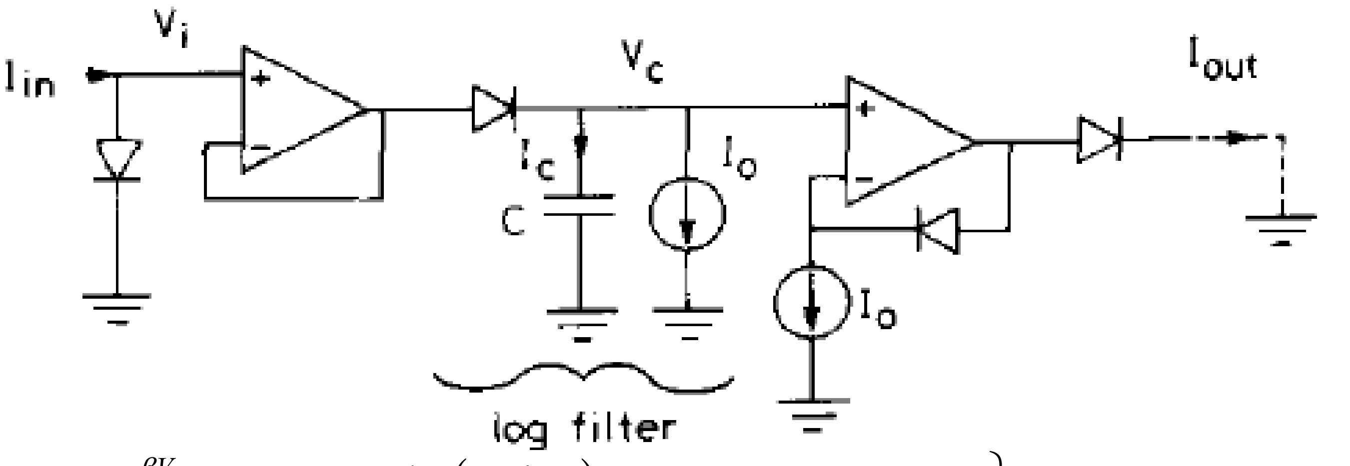
Log Domain filters

Origins from audio use of compression to overcome noise
First papers in that context

A case of **E**xternally **L**inear, **I**nternally **N**on-linear circuits:

However, in bipolar transistors I=exp(V), so there is the possibility an ELIN circuit is actually linear in the current mode!

The Adams Log domain filter



$$\left.\begin{aligned} &I_{in} = I_s e^{\beta V_i} \Rightarrow V_i = V_T \ln\left(I_i / I_s\right) \\ &I_C + I_0 = C\frac{dV_C}{dt} + I_0 = I_s e^{\beta\left(V_i - V_C\right)} = I_{in} e^{-\beta V_C} \\ &I_{out} = I_S e^{\beta V_1} = I_S e^{\beta V_C + \ln\left(I_0 / I_S\right)} = I_S e^{\beta V_C} I_0 / I_S = I_0 e^{\beta V_C} \end{aligned}\right\} \text{with } \beta = q / kT = 1 / V_T$$

$$\frac{dx}{dt} = \beta \frac{dV_C}{dt} x$$

$$C\frac{dV_C}{dt} + I_0 - I_{in} e^{-\beta V_C} = 0 \Rightarrow \frac{C}{\beta x}\frac{dx}{dt} + I_0 - \frac{I_{in}}{x} = 0 \Rightarrow \frac{C}{\beta}\frac{dx}{dt} + I_0 x = I_{in}$$

The Adams filter, continued

$$\frac{C}{\beta x}\frac{dx}{dt} + I_0 - \frac{I_{in}}{x} = 0 \Rightarrow \frac{C}{\beta}\frac{dx}{dt} + I_0 x = I_{in} \Rightarrow$$

$$\frac{dx}{dt} + \frac{\beta I_0}{C} x = \frac{\beta I_{in}}{C}$$

$$I_{out} = I_0 x$$

This is a first order low pass filter for x, with a Laplace space solution:

$$\frac{dI_{out}}{dt} + \frac{\beta}{C} I_0 I_{out} = \frac{\beta I_0 I_{in}}{C} \Rightarrow sx + \omega_0 x = \omega_0 u$$

$$\frac{X(s)}{U(s)} = \frac{\omega_0}{s + \omega_0} \ , \ \omega_0 = \frac{\beta I_0}{C}$$

This is in a “state variable” form, so we examine state variable models next

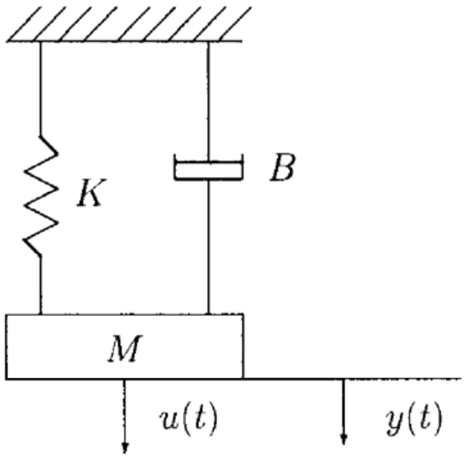
State-variable Models

- Introduction
- Standard State-variable Format
- Advantages of State-variable Models
- Simulation Diagrams
- State-variable Models from Transfer Functions
- Transfer Functions from State-variable Models

Introduction

- We have already come across:
  - **linear differential equation models**
  - **transfer function models**
- We now consider the **state-variable model**, also called the **state-space model**.
- This is a differential equation model, but the equation is written in a specific format.
- The idea is that an nth order differential equation is decomposed into a set of n 1st order equations written in matrix-vector form.
- This decomposition is achieved by defining internal variables, called **states**, whose time evolution completely characterises the behaviour of the system.

Example



$$M\ddot{y}(t) + B\dot{y}(t) + Ky(t) = u(t)$$

- Define $x_1(t) = y(t)$, $\quad x_2(t) = \dot{y}(t)$.

$$\Rightarrow \quad \dot{x}_1(t) = x_2(t)$$

$$\Rightarrow \quad \dot{x}_2(t) = -\frac{K}{M}x_1(t) - \frac{B}{M}x_2(t) + \frac{u(t)}{M}$$

or in matrix form:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{K}{M} & -\frac{B}{M} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

- This is an example of a **single input/single output** state-variable model.

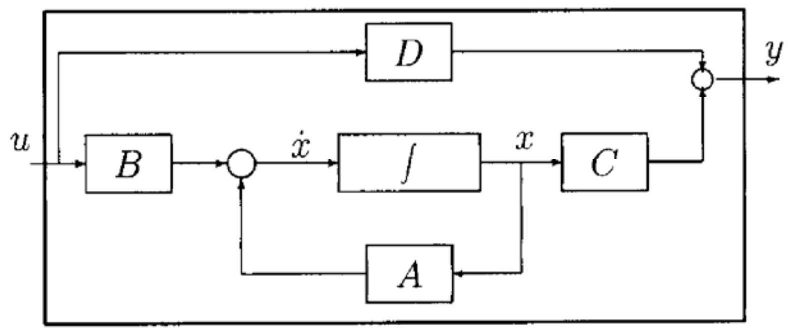
Standard State-variable Format

- **The state of a system at time t_1 is the amount of information at t_1 that, together with all inputs for $t \geq t_1$, uniquely determines the system behaviour for all $t \geq t_1$.**
- State-variable models have the form:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) && \textbf{(state equation)} \\ y(t) &= Cx(t) + Du(t) && \textbf{(output equation)} \\ x(0) &= x_0 && \textbf{(initial condition)} \end{aligned}$$

where

A	$n \times n$	**(system matrix)**
B	$n \times r$	**(input matrix)**
C	$p \times n$	**(output matrix)**
D	$p \times r$	**(direct feedthrough matrix)**
$x(t)$	$n \times 1$	**(state vector)**
$u(t)$	$r \times 1$	**(input vector)**
$y(t)$	$p \times 1$	**(output vector)**



Example

- Consider the coupled differential equations:

$$\begin{aligned} \ddot{y}_1 + k_1\dot{y}_1 + k_2 y_1 &= u_1 + k_3 u_2 \\ \dot{y}_2 + k_4 y_2 + k_5\dot{y}_1 &= k_6 u_1 \end{aligned}$$

- Define: $x_1 = y_1,\ x_2 = \dot{y}_1,\ x_3 = y_2$

- This gives the state equations:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -k_2 x_1 - k_1 x_2 + u_1 + k_3 u_2 \\ \dot{x}_3 &= -k_5 x_2 - k_4 x_3 + k_6 u_1 \end{aligned}$$

and the output equations:

$$y_1 = x_1, \qquad y_2 = x_3$$

- Collecting these into matrix form gives the 2-input 2-output state-variable model

$$\underbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix}}_{\dot{x}} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ -k_2 & -k_1 & 0 \\ 0 & -k_5 & -k_4 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{x} + \underbrace{\begin{bmatrix} 0 & 0 \\ 1 & k_3 \\ k_6 & 0 \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}}_{u}$$

$$\underbrace{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}}_{y} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{x}$$

Courtesy of Dr. Imad Jaimoukha

Advantages of State-variable Models

- State-variable models give knowledge about the **internal structure** as well as the input-output characteristics of the system.

- There are **computational** advantages:
  - **time-domain** matrix methods lend themselves naturally to computer solution, especially for high order models.
  - matrix methods enable us to easily determine the **transient response** and evaluate the system performance.

- A good unified framework for several advanced control theories, such as **optimal control** design methods.

- A natural framework for system **simulation**.

- Extensions to:
  - **multivariable** systems
  - **nonlinear** systems
  - **time-varying** systems

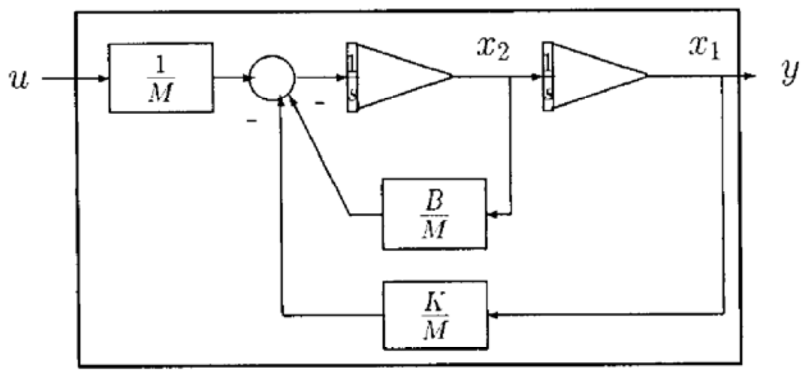
  are (relatively) straightforward.

Simulation Diagrams

- These are block diagrams (or signal flow graphs) that are constructed to have a given transfer function or to model a set of differential equations.
- There are three elements in a simulation diagram:
  - an **integrator** (whose transfer function is $\frac{1}{s}$)
  - a pure **gain**
  - a **summer**

  All these components can be easily constructed using simple electronic devices.
- They are useful in constructing computer simulations (digital or analogue) of a given system.
- For the mass-spring system we have:

$$\begin{aligned}\dot{x}_1(t) &= x_2(t)\\ \dot{x}_2(t) &= -\frac{K}{M}x_1(t) - \frac{B}{M}x_2(t) + \frac{u(t)}{M}\end{aligned}$$



State-variable Models from Transfer Functions

- There are simulation diagrams which can be derived from general transfer functions of the form:

$$\frac{y(s)}{u(s)} = g(s) = \frac{b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \cdots + b_0}{s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \cdots + a_0}$$

- Divide numerator and denominator by s^n:

$$y(s) = \frac{b_{n-1}s^{-1} + b_{n-2}s^{-2} + \cdots + b_0 s^{-n}}{1 + a_{n-1}s^{-1} + a_{n-2}s^{-2} + \cdots + a_0 s^{-n}}\, u(s)$$

- Set

$$e(s) := \frac{u(s)}{1 + a_{n-1}s^{-1} + a_{n-2}s^{-2} + \cdots + a_0 s^{-n}}$$

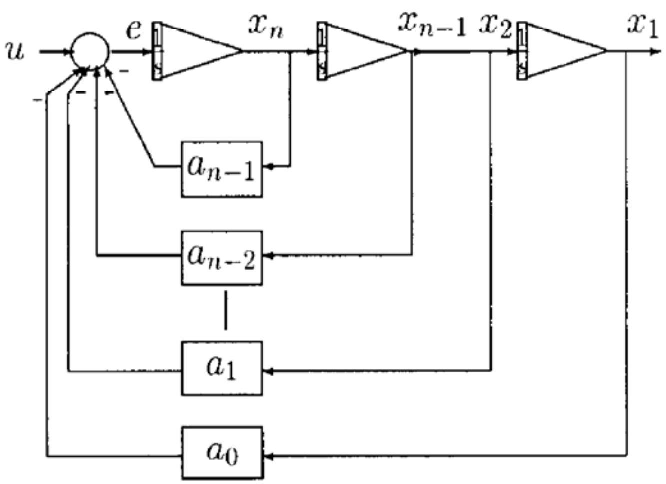
  so that

$$y(s) = [b_{n-1}s^{-1} + b_{n-2}s^{-2} + \cdots + b_0 s^{-n}]\; e(s)$$

- This gives

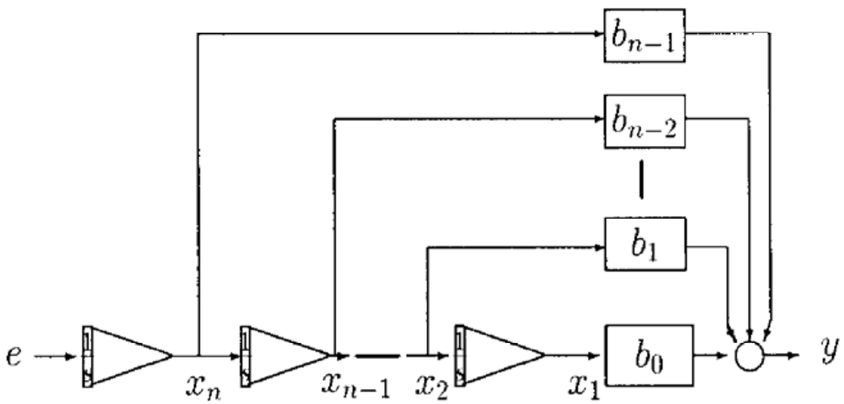
$$e(s) = u(s) - [a_{n-1}s^{-1} + a_{n-2}s^{-2} + \cdots + a_0 s^{-n}]\, e(s)$$

- This has the following simulation diagram:



- Complete the diagram by setting

$$y(s) = [b_{n-1}s^{-1} + b_{n-2}s^{-2} + \cdots + b_0 s^{-n}]\, e(s)$$



- The state variables satisfy:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ &\vdots \\ \dot{x}_{n-1} &= x_n \\ \dot{x}_n &= -a_0x_1 - a_1x_2 - \cdots - a_{n-2}x_{n-1} \\ &\quad -a_{n-1}x_n + u \end{aligned}$$

while the output is

$$y = b_0x_1 + b_1x_2 + \cdots + b_{n-2}x_{n-1} + b_{n-1}x_n$$

- In matrix form this yields the state-variable model:

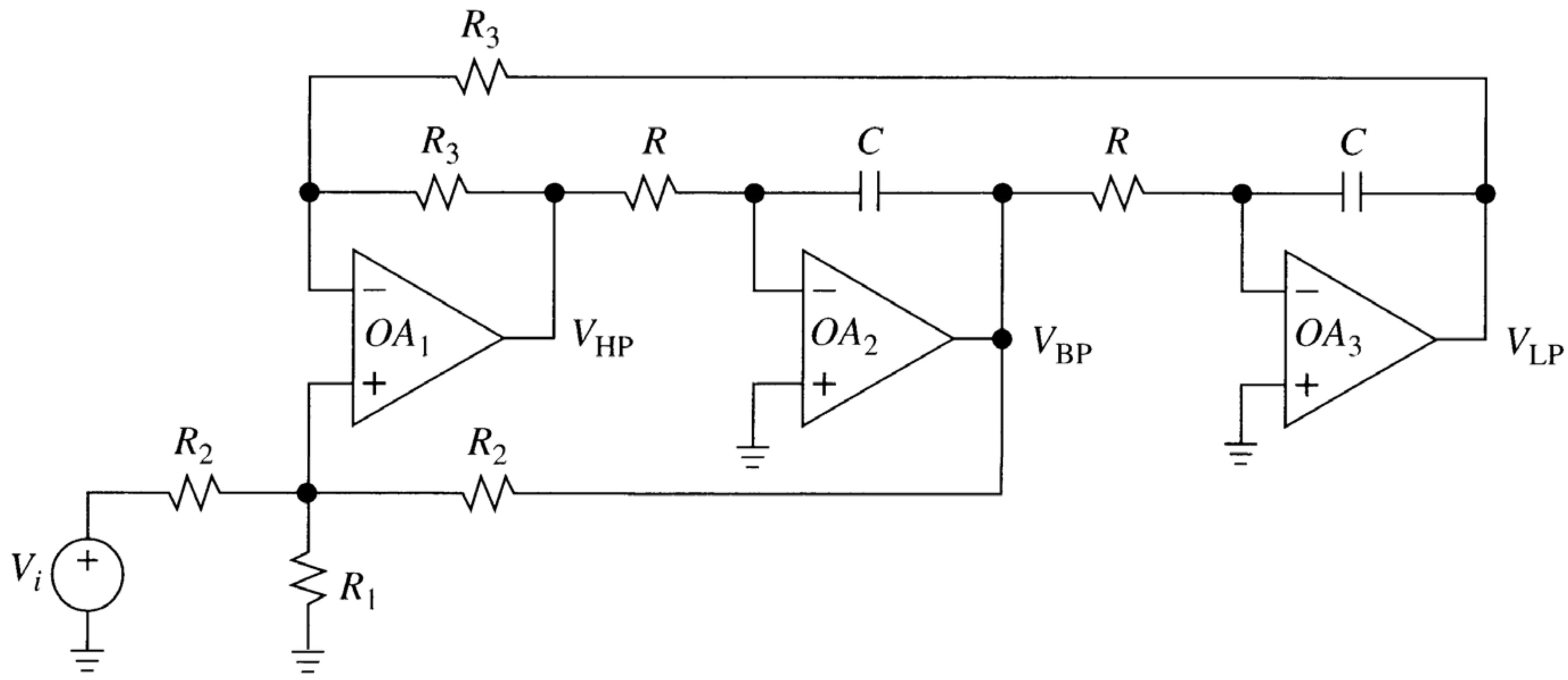
$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ -a_0 & -a_1 & \cdots & -a_{n-2} & -a_{n-1} \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} b_0 & b_1 & \cdots & b_{n-2} & b_{n-1} \end{bmatrix} x$$

- Note the direct connection with the coefficients of the transfer function:

$$g(s) = \frac{b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \cdots + b_0}{s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \cdots + a_0}$$

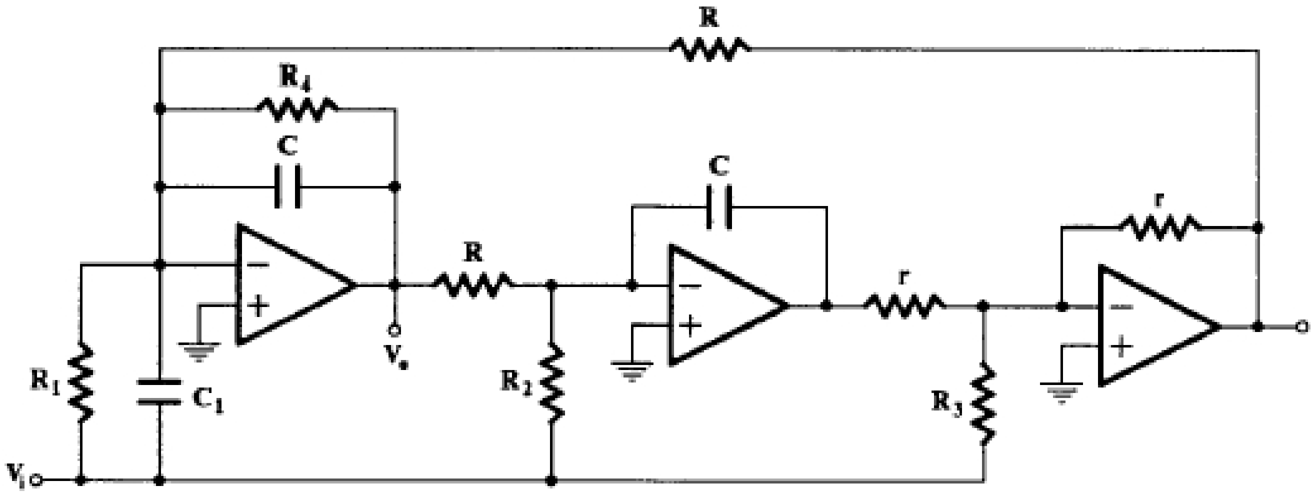
Example of voltage mode state variable filter: KHN



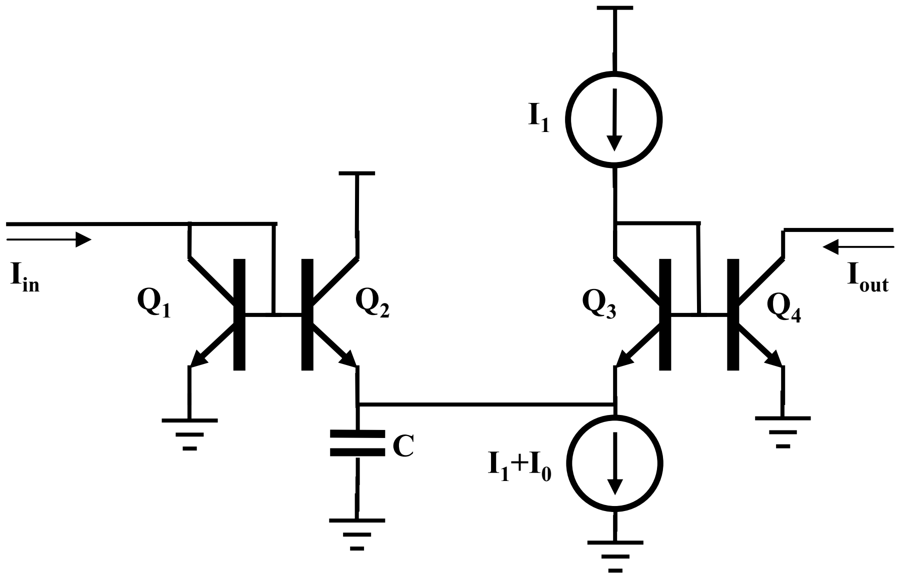
Observe the 3 possible outputs implementing low pass, band pass and high pass filters, demonstrating that each response is the derivative of the preceding one

Lossy state variable analysis: The Tow Thomas filter

- The state variable model is not unique. The decomposition can be slightly modified to work with lossy integrators (see Frey's paper)
- An example of a partial lossy decomposition is the Tow Thomas filter.

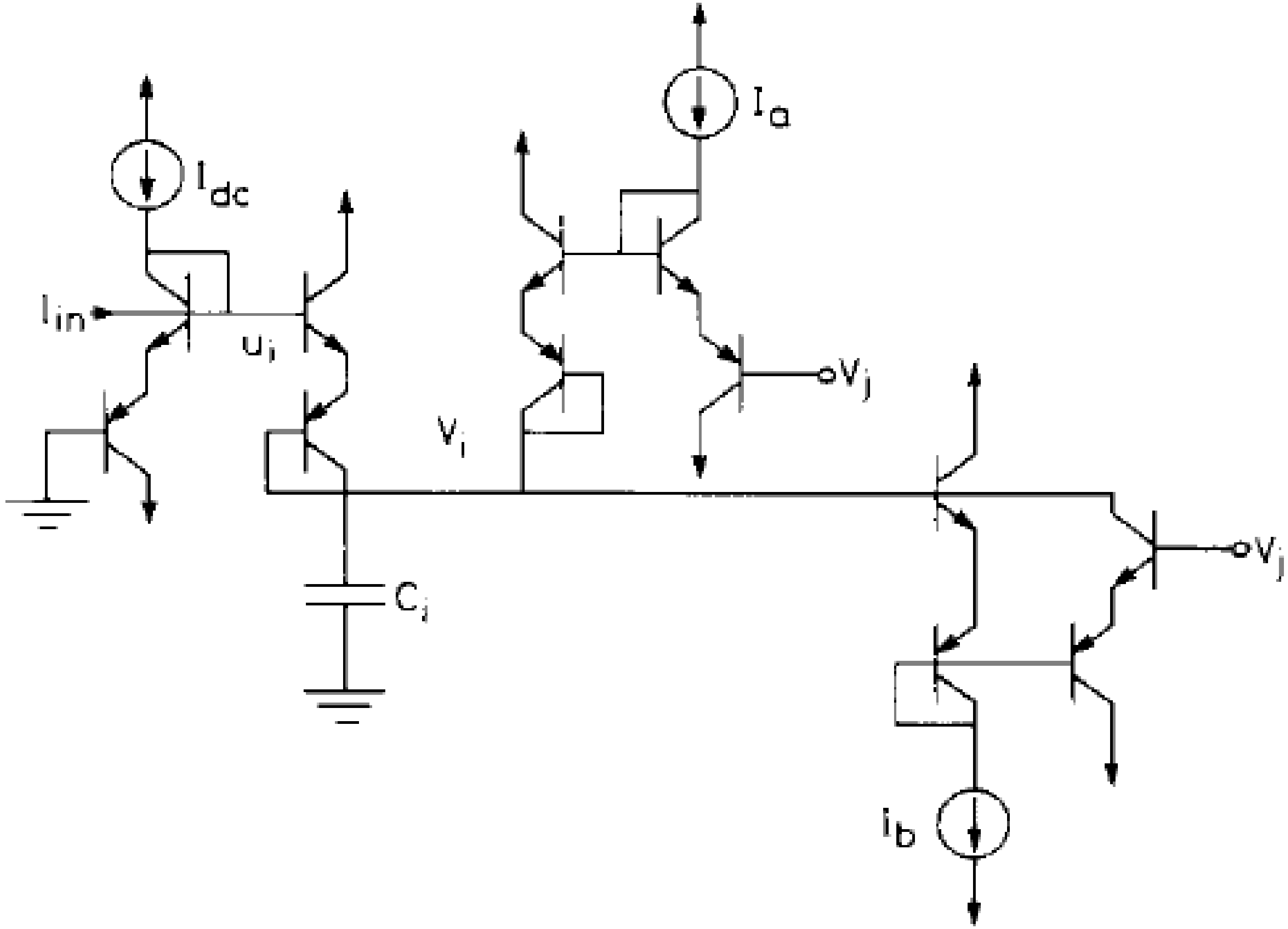


The Bernoulli cell

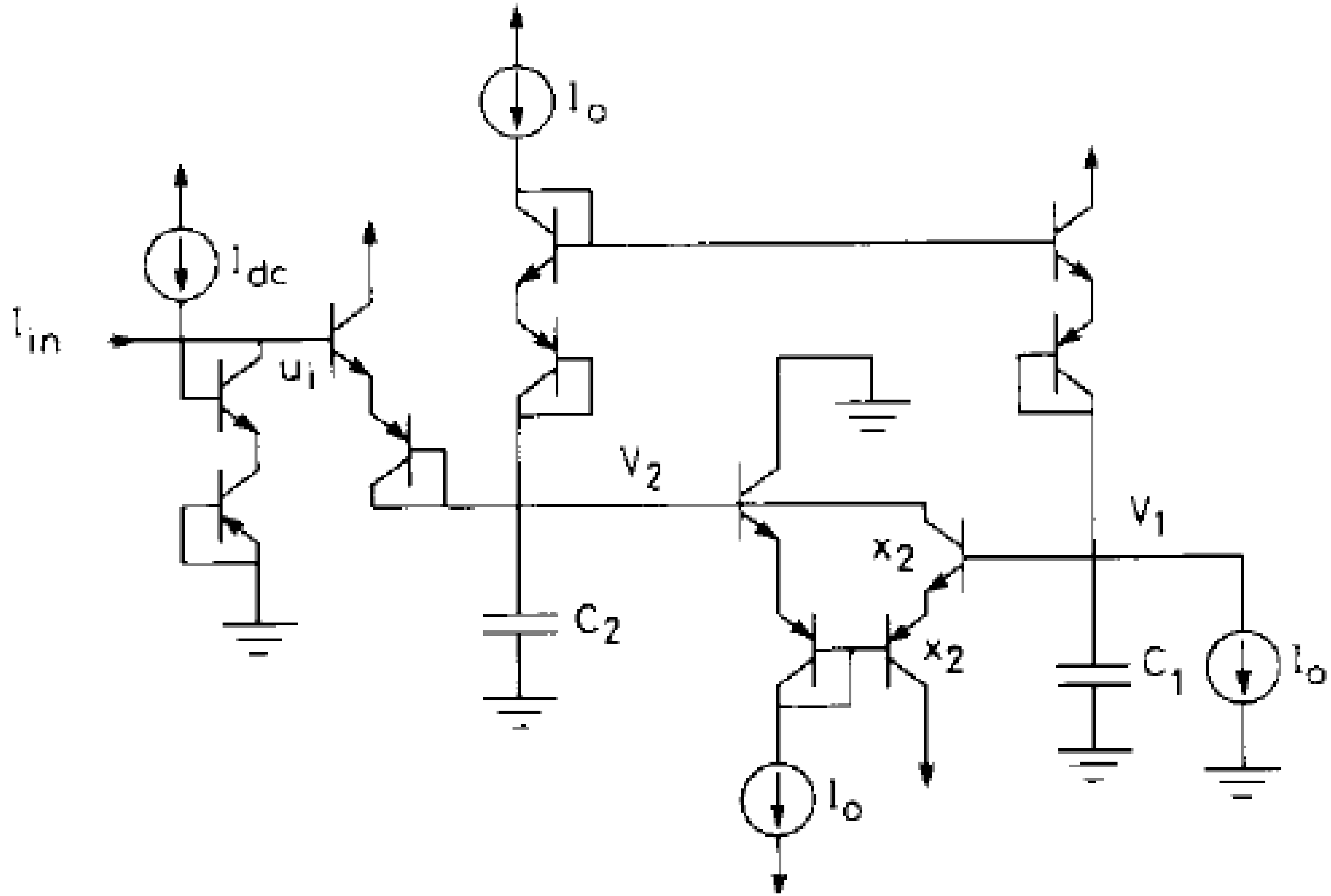


This is essentially the same circuit as the Adams circuit. Prove it!
If I_o is very small this can implement an essentially ideal integrator.

The Bernoulli cell using complementary devices



A biquad filter: 2 Bernoulli cells and a mirror linking them



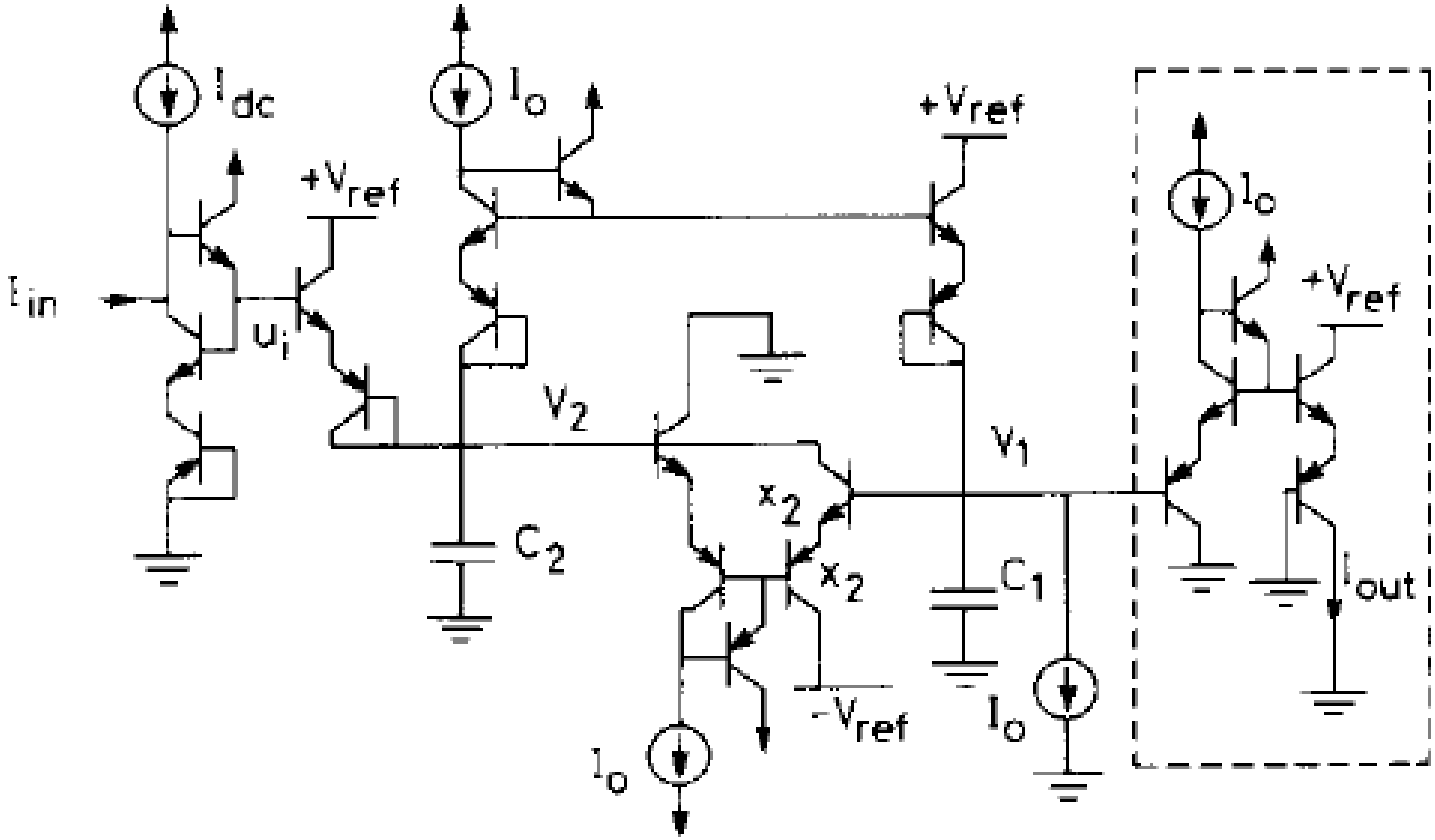


Fig. 7 *Complete second-order lowpass-filter schematic diagram*

Example of implementing the state variable output equation

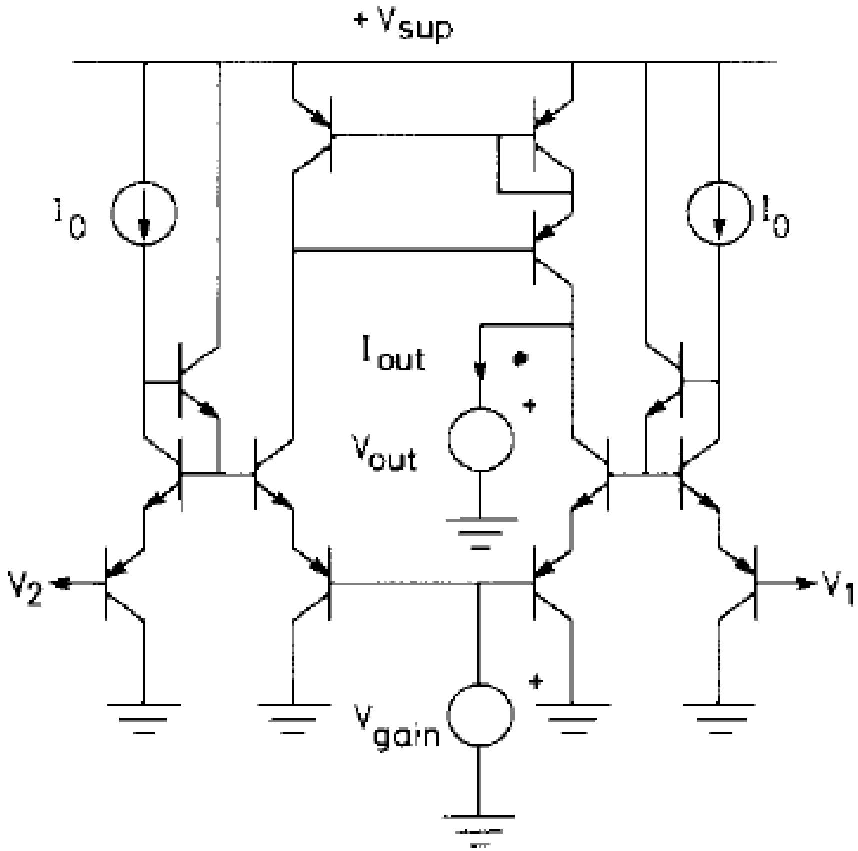


Fig. 8 *Output-circuit schematic diagram for converting the circuit of Fig 7 to a bandpass filter*

Implementations tend to have lower cutoff than designed

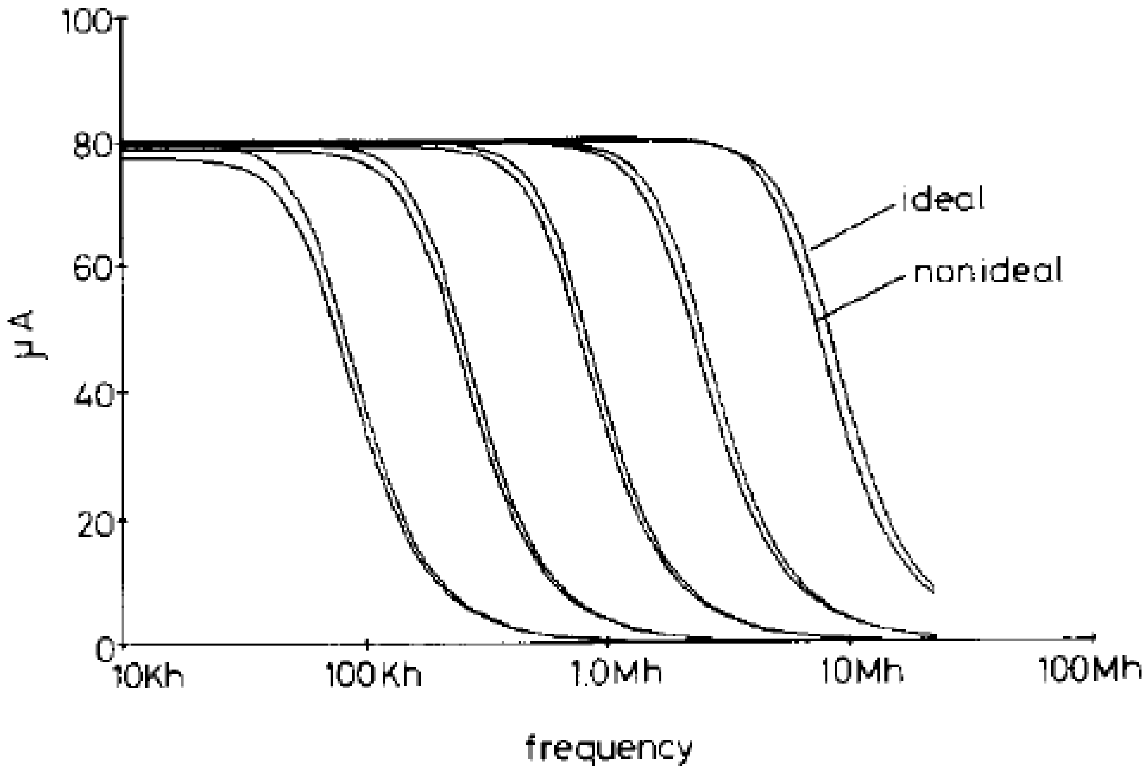


Fig. 9 *Second-order Butterworth lowpass log-filter characteristics*

Observe the current is not distorted!

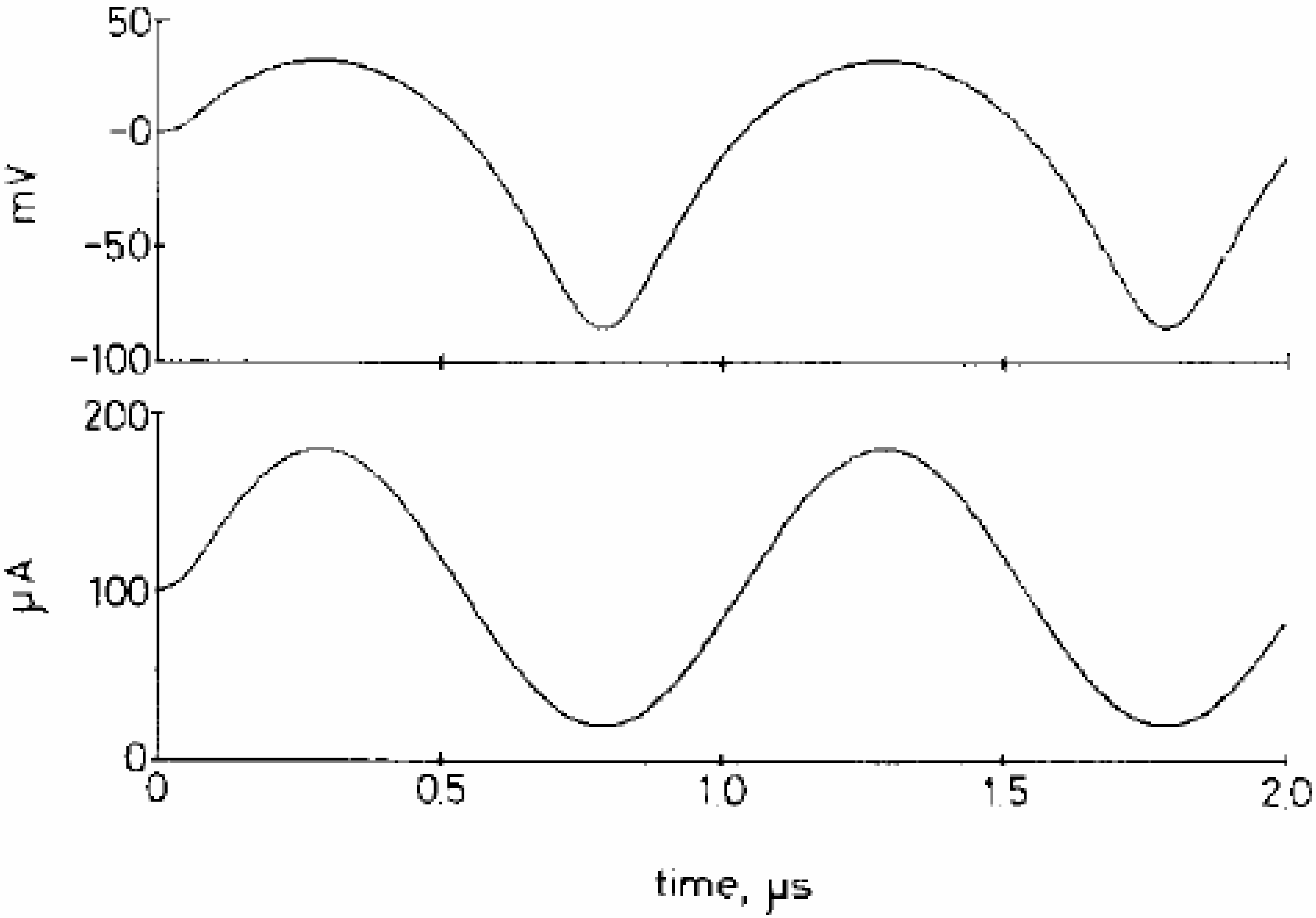


Fig. 10 *Zero-state response of second-order Butterworth filter*

Implementation tends to have higher Q than designed

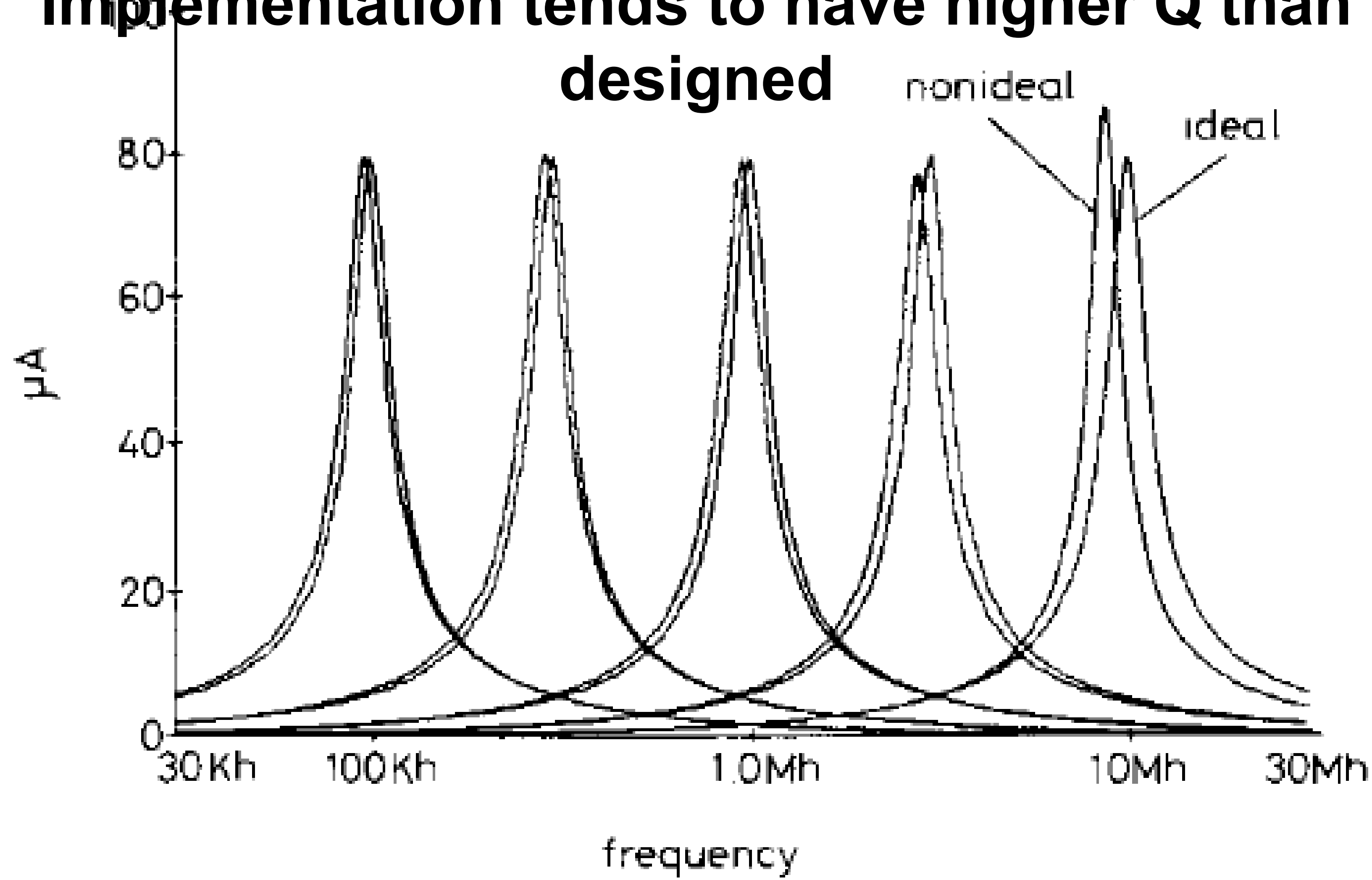
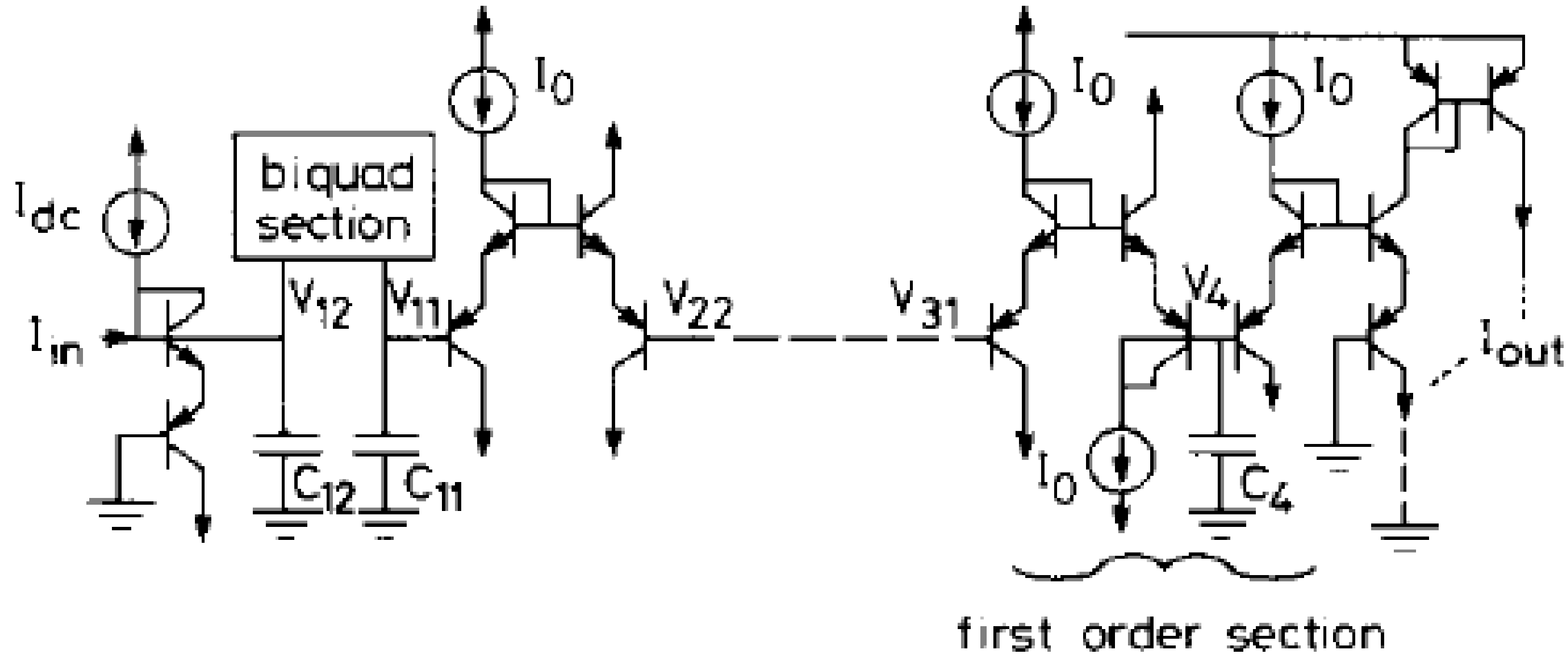


Fig. 11 *Bandpass log filter (Q = 5) characteristics*

Realising high order filters feedbck not shown



Note the linear Y scale, and the innacuracy in pole placement

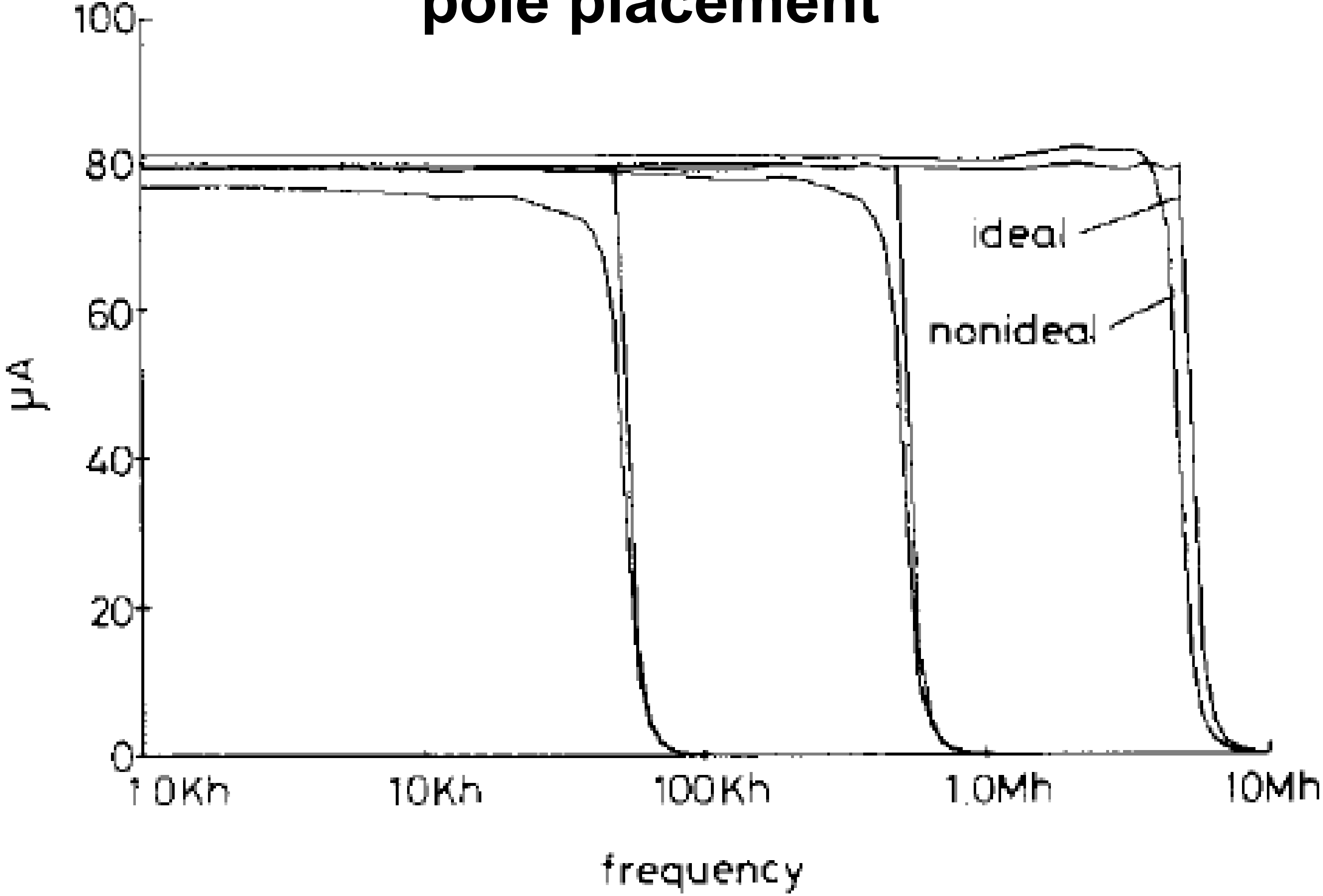
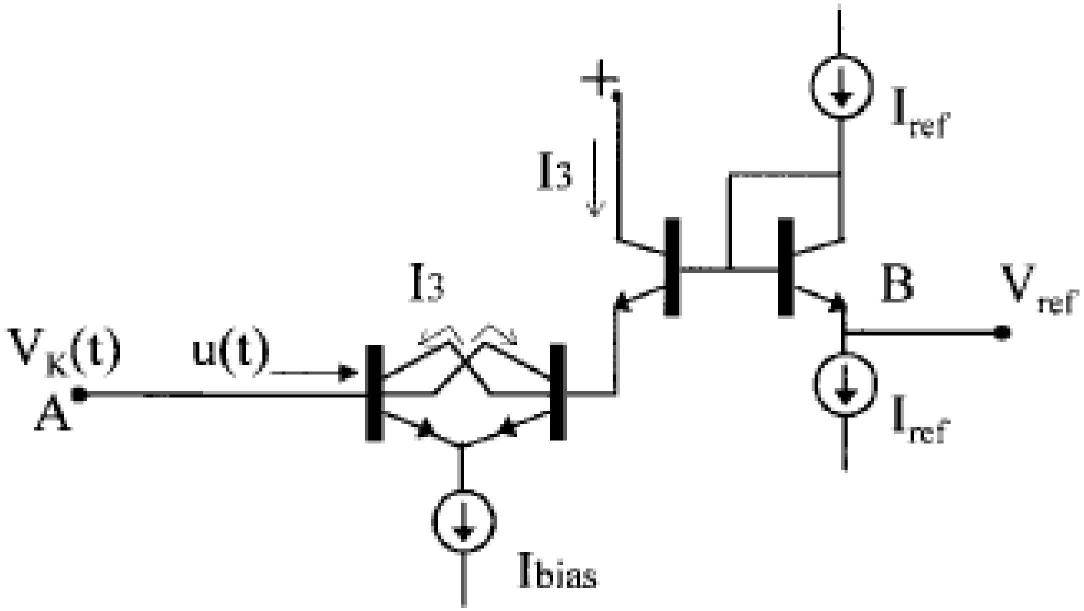


Fig. 14 *Seventh-order Chebychev log-filter characteristics*

ELIN ("E+" cell) Coupling structure INPUT current = exp (V difference)



(a)

$$u(t) = I_{\rm ref}\ \exp[(V_{\rm ref} - V_K(t))/V_T].$$

Example of Biquad analysis