2 Impedance matching-discrete

Objectives

- Perform impedance matching using capacitors and inductors, at a desired frequency and bandwidth.
- Calculate losses in a matching network

Motivation

We concluded the discussion of antennas, by mentioning the noise temperature of a receiver front end. We found that:

$$T_{system} = T_{ant} + \frac{1}{k_c} T_{rec} \tag{1}$$

Where k_c was the efficiency of the antenna-receiver link, or the ratio of power transferred to power lost in heat. To minimise system temperature, all other factors kept constant, we have to maximise k_c , i.e. maximise the ratio of the power transferred from the antenna to the preamplifier to the noise power.

Available Power

An ideal voltage (current) source can deliver an infinite amount of power to a load:

$$max(P_{deliv}) = max(Re(I^*V)) = V^2/R_L \tag{2}$$

$$max(P_{deliv}) = max(Re(I^*V)) = I^2/G_L \tag{3}$$

As $R_L \to 0$ for the voltage source or $G_L \to 0$ for the current source, $P_{deliv} \to \infty$.
The arguments for ideal voltage and current sources are symmetric, so we often talk of the *immittance* to avoid repeating an argument about resistances and voltages for its dual case of admittances and currents. The equivalence between Thevenin and Norton descriptions of a circuit is the simplest example of this symmetry.
An ideal source, therefore, cannot physically be realised. It follows that every real source must have a finite immittance, and that the distinction between current and voltage sources depends on whether the source impedance is large or small compared to a load. Clearly the distinction is often quite arbitrary and the same source connected to different loads can behave as a voltage or current source depending on the value of the load. To make the present argument concrete, we talk about a Thevenin source, i.e. a voltage source V_0 (RMS) with an output impedance Z_s ($fig.1$).This source can deliver a maximum amount of power to a conjugate load, $Z_L = Z_s^*$.

$$P_a = \frac{V_0^2}{4Re(Z_s)} \tag{4}$$

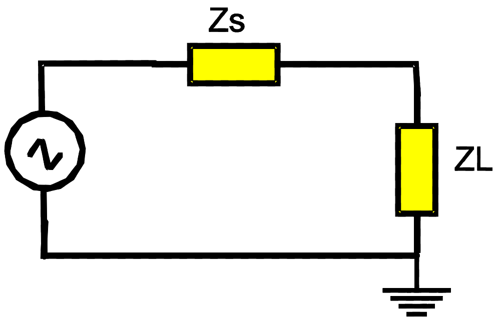

Figure 1: The power delivery problem. Maximum power transfer occurs when $Z_L = Z_s^*$.

This is called the *Available Power* of this source. Please bear in mind that the "source" may well be the output of a circuit. Usually, we need to drive loads whose immittance is not the complex conjugate of the source immittance. At high signal levels, lower frequencies, and in many wideband applications, we are willing to incur the power transfer (i.e. the noise budget!) penalty of mismatch. At receiver front ends, and near the frequency operating limits of active devices, mismatch is not an option, and a *transformer* is introduced between R_s and R_L. The transformer is usually a lossless (actually low loss) network, which maps R_L into an apparent impedance $Z_a \simeq Z_s^*$.
The power transferred to the load is:

$$P = \mathrm{Re}(IV^*) = \mathrm{Re}(I^*V) = P_a \frac{4\,\mathrm{Re}(Z_a)\,\mathrm{Re}(R_s)}{\mathrm{Re}(Z_a + Z_s)^2 + \mathrm{Im}(Z_a + Z_s)^2} = P_a M \tag{5}$$

This is the same "mismatch factor" we mentioned in chapter 1, in the context of the power radiated by an antenna.
A fundamental theorem of microwave electronics is that *"the mismatch factor is constant along a lossless connection"*, i.e. the same regardless of which node it is calculated on.
Proof: If it were not, then some power would have to be dissipated in the presumed lossless network.

L-C matching

"L" Network

The simplest form of a matching transformer consists of an LC network, and is called the "L matching network"(figure 2).
Assume we need to match a Z_L to a Z_s, such that $Re(Z_s) = R_s$, $Re(Z_L) = R_L$. There is no loss of generality to assume that both Z_s and Z_L are real, i.e. $Z_s = R_s$ and $Z_L = R_L$.

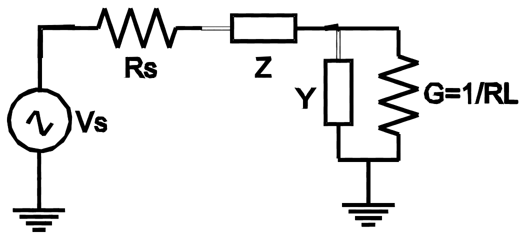

Figure 2: "L" Matching network, $R_s < R_L$. X and Y can be either Capacitor-Inductor, or Inductor-Capacitor.

If they are not real, we can always precede the procedure we describe here by cancelling ("tuning out") the imaginary parts of Z_s and Z_L, each with a single, capacitor or inductor. We will also assume $R_s < R_L$, since the argument for $R_s > R_L$ is essentially identical except for a port reversal. Assuming we have suceeded in losslessly matching source to load by use of a Z in series to the source and a Y in parallel to the load (fig. 2), the invariance of the mismatch factor theorem assures us that the series and parallel branches are matched, so:

$$Z_s = Z_p^* \Rightarrow R_s + Z = \frac{1}{G + Y^*} \tag{6}$$

The network is lossless, therefore, both Z and Y are purely imaginary: $Z = jX$ and $Y = jB$. Therefore:

$$R_s + jX = \frac{1}{G - jB} = \frac{G + jB}{G^2 + B^2} \Rightarrow \tag{7}$$

$$G = R_s(G^2 + B^2) \tag{8}$$

$$B = X(G^2 + B^2) \tag{9}$$

and we can write

$$\frac{B}{G} = \frac{X}{R_s} = \pm Q_L \tag{10}$$

It is easy to show that the *Loaded Quality factor ("Loaded Q")* Q_L is given by:

$$Q_L = \sqrt{\frac{R_L}{R_s} - 1} \tag{11}$$

To summarise, the matching network consists of a series reactor to the smallest resistance (R_s in this case), of magnitude:

$$Z = jX = \pm jQ_L R_s \tag{12}$$

and a parallel susceptor to the smallest admittance ($G_2 = 1/R_2$ in this case), of magnitude:

$$Y = jB = \pm jQ_L G = \pm j\frac{Q_L}{R_L} \tag{13}$$

Both X and B are calculated at the desired frequency for perfect matching, and each can be either a capacitor or inductor, as long as X and B have the same sign.
The circuit we have constructed is resonant at a frequency:

$$\begin{aligned} f_0 &= \frac{1}{2\pi\sqrt{LC}}\sqrt{\frac{Q^2}{Q^2+1}} \\ \lim_{Q\to\infty} f_0 &= \frac{1}{2\pi\sqrt{LC}} \end{aligned} \tag{14}$$

If Q_L is large, the matching network has a 3dB (power) bandwidth of

$$BW = \Delta f \simeq \frac{f_0}{Q_L} \tag{15}$$

An interesting quantity is the ratio of peak voltage through the capacitor to peak current through the inductor. It is a quantity to keep in mind since peak voltages and/or currents in matching networks can easily be excessive.

$$Z_0 = \frac{V_C}{i_L} = \sqrt{\frac{L}{C}} = R_s\sqrt{Q^2+1} = \frac{1}{G\sqrt{Q^2+1}} = \sqrt{\frac{R_s}{G}} \tag{16}$$

This ratio is the equivalent of the *characteristic impedance* Z_0 of distributed networks.
Example Design an L network to match a 250 ohm antenna to a 50 ohm cable at $f = 159.2$ MHz..
The Q of this matching problem is:
$Q = \sqrt{5-1} = 2$. $\omega_0 = 2\pi \times 159 \times 10^6 = 10^9$
Let's choose X to be an inductor and B a capacitor:
$X = \omega_0 L = QR_s = Q \times 50 \Rightarrow L = 100nH$
$B = \omega_0 C = QG = Q/250 \Rightarrow C = 8pF$
The circuit resonates, as designed, at

$$f_0 = \frac{1}{2\pi\sqrt{LC}}\sqrt{\frac{Q^2}{Q^2+1}} = 1.118 \times 10^9\sqrt{\frac{4}{5}} = 1Grad/s \tag{17}$$

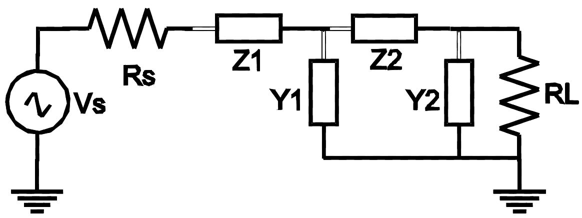

Figure 3: Example of a cascade of 2 L matching stages. For effective bandwidth see text

An "L" matching network has its bandwidth uniquely determined by the two impedances it is matching. A smaller Q can always be achieved by cascading several "L" sections (figure 3). In such a case we perform the sequence of matching steps:

$$R_s \rightarrow R_s \left(\frac{R_L}{R_s}\right)^{\frac{1}{n}} \rightarrow R_s \left(\frac{R_L}{R_s}\right)^{\frac{2}{n}} \rightarrow \ldots R_L$$

And the Q of each section is:

$$Q^{(n)} = \sqrt{\left(\frac{R_L}{R_s}\right)^{\frac{1}{n}} - 1} \tag{18}$$

The normalised bandwidth of the matching net is:

$$BW = \frac{f_0}{Q^{(n)}}\sqrt{\sqrt[n]{2} - 1} \tag{19}$$

Where we have made use of the expression for the bandwidth of a cascade of tuned sections:

$$BW_n = BW_1\sqrt{\sqrt[n]{2} - 1} \tag{20}$$

in the limit of a large number of sections, the total bandwidth approaches a number of order unity, namely,

$$lim_{n\rightarrow\infty} BW = \sqrt{\frac{\ln(2)}{\ln(m)}} \tag{21}$$

where m the ratio of the impedances we are matching. This cascade of L networks is the prototype of a useful distributed wideband matching network, the *Stepped Transformer*

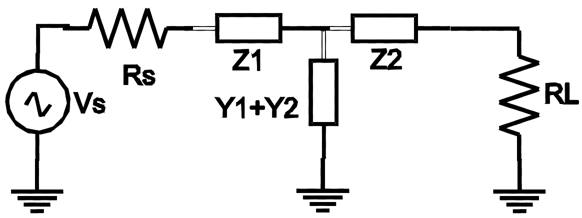

Figure 4: Example of a tee network. Y1 and Y2 add admittances. Normaly Y1+Y2 is implemented by a single component.

where a long ($l \gg \lambda$) section of a transmission line, of characteristic impedance smoothly varying from R_s to $\frac{1}{G}$ performs the impedance transformation. The fractional bandwidth of such a transformer is known to be of the order of unity.
Example How many stages are required to make a wideband ($Q \simeq 1$) match between a 300 ohm antenna and a 50 ohm receiver? What are the impedance matching problems generated?
A single L section cannot be used because this would have $Q = \sqrt{300/50 - 1} = \sqrt{5} = 2.24$. To lower the Q we need a cascade of L sections, each with a Q at most equal to the specification (better a little lower Q to compensate for the bandwidth narrowing in a cascade. This implies that for each stage $Q \leqslant 1 \Rightarrow R_{hi}/R_{lo} \leqslant 2$. Then a minimum number of stages would be: $n \geq \log_2 6$. A suitable choice is n=3. The impedance matching to be solved are: 50 to 90.86, 90.86 to 165.1 and 165.1 to 300 ohm. The q of all three is $\sqrt{\sqrt[3]{6} - 1} = 0.904$.

Tee and Pi networks

The technique just illustrated achieves wider bandwidths than the single section L network. Often, narrower bandwidths are desired. In this case we can match *both* terminal impedances Z_s and Z_L to a common impedance higher than either, in which case we construct a Tee network, or we can match both impedances to a smaller than either common impedance, in which case we talk about a Pi circuit. Both allow us to select the frequency and the bandwidth at the same time. Let's see both of them in some detail.
A Tee network (figure 4) matches both R_s and R_L to a common intermediate impedance R_I. Since Z_1 and Z_2 are both in series with the resistances we wish to match, it follows that

$$R_I > R_L > R_S \tag{22}$$

and that

$$Q \simeq Q_{max} = \sqrt{\frac{R_I}{R_s} - 1} \tag{23}$$

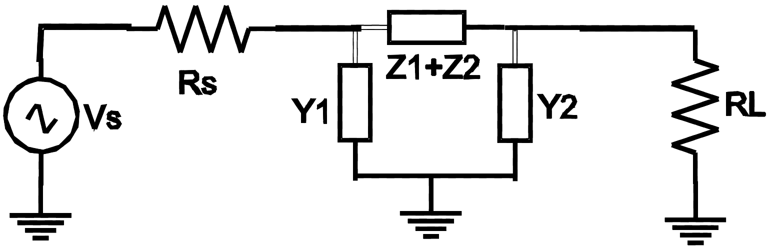

Figure 5: Example of a Pi matching network. $Z_1$and Z_2 are implemented by a single component.

Clearly a tee network always has a smaller bandwidth than a simple L network.

A Pi network (figure 5), on the other hand, matches both R_s and R_L to a common intermediate impedance:

$$R_I < R_s < R_L \tag{24}$$

Again, the Q is the maximum Q of the two L networks, so the Pi network is also a narrower bandwidth approach. Note an interesting simplification. Both Tee and Pi networks are implemented with three elements, the central one having the sum reactance or susceptance at resonance.

An obvious generalisation is a repetition of, say, tee networks keeping the type of the series impedance the same. In the limit of an infinite sequence of Tee (or Pi) networks we get a uniform transmission line. (figure 6)

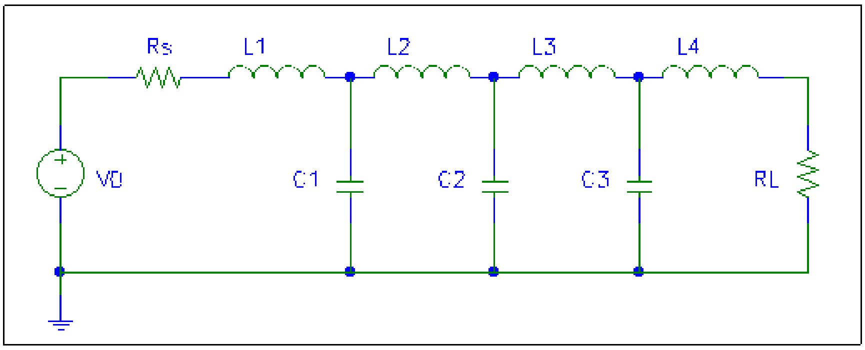

Figure 6: A repetition of Tee matching sections results to a distributed circuit closely resembling a transmission line.

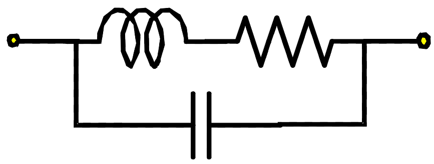

Figure 7: Simplified equivalent circuit of a physical inductor. It is clear that above the resonant frequency the component is actually capacitive.

Example We decide to make a 15.9 MHz bandwidth match between a 75 ohm antenna and a 50 ohm receiver at 159MHz..
We will chose to do it with a Pi network, so we will be raising the impedance level. The required maximum Q is:

$$Q_L \simeq \frac{f_0}{BW} = 10 \Rightarrow \frac{R_I}{R_{min}} = Q^2 + 1 = 101 \tag{25}$$

therefore we raise the impedance level for the common match to

$$R_I = 5050 \tag{26}$$

The problems of matching 50 and 75 ohms to 5050 ohms are left as an exercise.

Losses

Thus far we have only dealt with the *loaded Q* of a matching network, and have assumed that the components are ideal, i.e resistors are described by a resistance value, inductors by an inductance, and capacitors by a capacitance. At lower frequencies this is an accurate approximation, as we can neglect lead inductances, capacitance between component terminals, resistance of wires and dielectric losses in capacitor dielectrics. Real components at radio frequencies must be described by elaborate models, derived either by plausibility -handwaving!- arguments, or by electromagnetic field simulation. Often a simplified empirical approach is adopted, in which capacitors and inductors have both a quality factor and a resonance frequency. Above this resonance frequency the component reverses behaviour. That is, an inductor exhibits a capacitive behaviour, while a capacitor exhibits an inductive behaviour.
An approximate equivalent circuit for a physical inductor is in fig. 7.
This network looks very much like the L matching network we discussed before. Without further proof, the *Unloaded* Q of the inductor is:

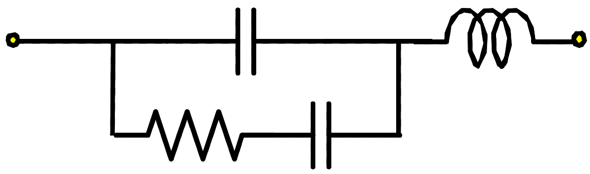

Figure 8: Simplified high frequency equivalent of a physical capacitor. The parallel branch resistance is due mostly to dielectric losses. A series terminal resistance loss has not been included.

$$Q_u \triangleq \frac{\omega_0 L}{R} \tag{27}$$

with the resonant frequency itself depending on Q, especially at low Q:

$$\omega_0 = \frac{1}{\sqrt{LC}}\sqrt{\frac{Q^2}{Q^2+1}} \simeq \frac{1}{\sqrt{LC}}\left(1 - \frac{R^2 C}{2L}\right) \tag{28}$$

A physical capacitor behaves like the circuit in fig 8, and its unloaded Q_u is usually large. The efficiency of a matching network constructed with components of a given unloaded Q is approximately:

$$\eta = \frac{P_{deliv}}{P_{avail}} \simeq \frac{1}{1 + \frac{Q_L}{Q_u}} \tag{29}$$

As long as we are designing for a loaded Q much smaller than the unloaded Q of the available components the efficiency will be quite high.

Example The peak efficiency of the matching net in the first example, that of matching a 250 ohm antenna to a 50 ohm cable at $f = 159.2$ MHz., if the inductor resistance is 10 Ohms is:

$$Q_u \triangleq \frac{\omega_0 L}{R} = \frac{10^9 \times 100 \times 10^{-9}}{10} = 10 \tag{30}$$

therefore,

$$\eta = \frac{1}{1 + \frac{Q_L}{Q_u}} \simeq \frac{1}{1 + \frac{2}{10}} = \frac{10}{12} = 83\% \tag{31}$$

This is not an unrealistic efficiency for this circuit. It may well be that such an inductor has its own resonance close to the 150 MHz operating frequency, in which case its full impedance must be taken into account. If its self resonance is *below* the operating frequency, then the component is useless.

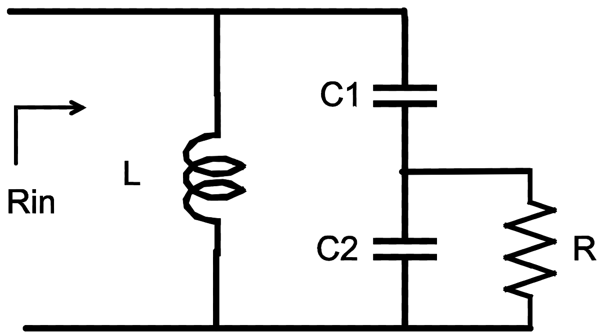

Figure 9: Tapped capacitance transformer. Normally the capacitive voltage divider C_1 C_2 can be considered perfect, and the conductance of R negligible compared to the admittance of C_2.

Exercises

1. Analyse the function of the "tapped capacitance transformer" (fig. 9). This circuit is commonly used in oscillators serving both as a resonator and a transformer.
2. Do the same for the transformer obtained by replacing capacitors with inductors, and vice versa, in the previous exercise.