Preconditioners for Inexact Interior Point Methods for Predictive Control

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***Abstract*— This paper presents a new method for solving a linear discrete-time finite horizon optimal control problem (FHOCP) with quadratic cost and linear constraints on the states and inputs. Such a FHOCP needs to be solved online, at each sampling instant, in predictive control. In order to solve such a FHOCP, it is necessary to solve a quadratic programming (QP) problem. The proposed technique uses an inexact interior-point method (IIPM) to solve the QP problem. This new technique is computationally more efficient than the Riccati Recursion method of Rao, Wright and Rawlings (Journal of Optimization Theory and Applications, 1998), when measured in terms of the number of floating point operations. The computational advantage is obtained by the use of an inexact Newton method, and with the use of novel preconditioners in the minimum residual (MINRES) method. The computational performance of this method is demonstrated by numerical results.**

I. INTRODUCTION

Predictive control has been used successfully for plants where the process dynamics is slow [1]. However, recently attempts have been made to show its efficacy in fast processes, such as aircraft and diesel engine, facilitated by the rapid growth in computing power [2], [3]. The problem is to find, over a finite horizon, a sequence of optimal control inputs for a linear discrete-time system with quadratic cost and mixed inequality constraints on the states and control inputs. At each sampling time, only the first control input is applied to the plant. At the next time step, the same procedure is repeated using the new state of the plant. The aim of this work is to reduce the computational cost of solving the resulting finite horizon optimal control problem.

To extend the use of predictive control to fast systems, two approaches exist; in the first approach, a large number of QPs are solved off-line for all possible initial states of the plant, then an explicit function or lookup table is formed. The online control computation then just reduces to the simple evaluation of that explicit function [4]. The major drawback in this approach is that the number of entries in the explicit function grows exponentially with the horizon length, number of states and control inputs. In the second approach a QP problem is solved online [5]. This approach does not have the limitation on small number of states and inputs. In this paper, we adopted this approach.

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In large-scale optimization problems inexact-Newton methods have been proposed to reduce the computational effort in the solution of the nonlinear optimality conditions [6], [7]. The main idea of this method is to terminate the linear solver early with less accuracy in the initial iterations when we are far from the optimal solution. In this paper, we investigate the efficacy of these methods in solving a finite horizon optimal control problem (FHOCP).

Our focus on iterative methods, such as the minimum residual (MINRES) method [8], is mainly due to two reasons [9]. Firstly, iterative methods have a higher ratio of addition and multiplication operations to division and square root operations, compared to direct methods, hence are more efficient from a hardware point of view. Secondly, in iterative methods, we can trade-off accuracy with computational time.

The main contribution of this paper is the reduction of the computational cost in the solution of the FHOCP with the development of new and effective preconditioners. The second contribution is the construction of a low-cost, approximate coefficient matrix appearing in the linear system in an inexact interior-point method (IIPM), which reduces the computational cost, and provides a perturbed linear system with low rank perturbation matrix. The final contribution is the development of a result that states that the solution of this perturbed linear system, with a perturbation matrix of rank r_Δ, can be computed using a preconditioned MINRES method in at most $r_\Delta + 1$ iterations.

Notation: For a matrix $A > 0$ (≥ 0) means that A is positive (semi-positive) definite and for vector $x < 0\,(\leq 0)$ means that each element of x is negative (non-positive). For a set X, card(X) denotes the number of elements in X. The symbol $\otimes$ represents the Kronecker product.

II. PROBLEM DESCRIPTION

Consider a discrete-time linear time-invariant system of the form

$$x_{i+1} = Ax_i + Bu_i, \tag{1}$$

where $x_i \in \mathbf{R}^n$ is the state vector and $u_i \in \mathbf{R}^m$ is the input vector at the i-th time instant. Let $x = x_0 \in \mathbf{R}^n$ be the measurement or estimate of the state at the current time instant. The objective is to find, over a finite horizon of length N, a sequence of optimal control inputs $u_0, \ldots, u_{N-1}$ subject to equality constraints (1) and the inequality constraints

$$J_i x_i + E_i u_i \leq d_i, \quad i = 0, \ldots, N-1, \tag{2a}$$

$$J_N x_N \leq d_N, \tag{2b}$$

while minimizing the following quadratic objective function

$$x_N{}^T P x_N + \sum_{i=0}^{N-1} \left(x_i{}^T Q x_i + u_i^T R u_i + 2 x_i^T M u_i \right) \quad (3)$$

with $R > 0$, $Q - MR^{-1}M^T \geq 0$ and $P \geq 0$, $J_i, J_N \in \mathbf{R}^{l \times n}, E_i \in \mathbf{R}^{l \times m}, Q, P \in \mathbf{R}^{n \times n}, R \in \mathbf{R}^{m \times m}, M \in \mathbf{R}^{n \times m}$.

To solve the above problem the vector of unknowns is defined as $\theta := \begin{bmatrix} x_0^T & u_0^T & x_1^T & u_1^T & \cdots & u_{N-1}^T & x_N^T \end{bmatrix}^T$.

The above problem can be converted to a QP problem, which can be defined as:

$$\min_{\theta} \frac{1}{2} \theta^T H \theta \quad \text{subject to} \quad F\theta = f(x), \quad G\theta \leq d, \quad (4)$$

where $\theta \in \mathbf{R}^{n_\theta}, H \in \mathbf{R}^{n_\theta \times n_\theta}, F \in \mathbf{R}^{n_e \times n_\theta}, G \in \mathbf{R}^{n_i \times n_\theta}$ with $n_\theta = (n+m)N+n$, $n_e = n(N+1)$ and $n_i = l(N+1)$ is the number of inequality constraints. The matrices H, F, G and the vectors f, d are defined in the Appendix. To simplify the notation, the (x) is omitted in vector $f(x)$.

To solve the QP problem two approaches are used, namely active set methods [10] and interior point methods (IPMs) [11], [5]. We focus on IPMs because they have polynomial computational complexity while active set methods have exponential complexity in the worst case [11].

III. INTERIOR POINT METHODS

In this section we review the ideas in interior point methods [11]. The Karush-Kuhn-Tucker (KKT) conditions of (4) are

$$\begin{aligned} H\theta + F^T\nu + G^T\phi &= 0, && (5a) \\ F\theta - f &= 0, && (5b) \\ -G\theta + d - s &= 0, && (5c) \\ \Phi S \mathbf{1}_{n_i} &= 0, \quad \phi, s \geq 0, && (5d) \end{aligned}$$

where $\nu \in \mathbf{R}^{n_e}$ and $\phi \in \mathbf{R}^{n_i}$ are called dual variables, $s \in \mathbf{R}^{n_i}$ is a vector of slack variables, $\mathbf{1}_{n_i} \in \mathbf{R}^{n_i}$ is a vector of ones and Φ, S are diagonal matrices defined by $\Phi := \text{diag}(\phi), \quad S := \text{diag}(s)$.

An IPM which uses an inexact Newton method to solve the nonlinear equations (5) is called an inexact interior point method. An infeasible and inexact IPM is described in Algorithm 1 for a QP problem. This is an extension of the infeasible-path-following IPM (Algorithm IPF) of [11], which was designed for a linear programming problem.

We refer to the while loop of Algorithm 1 as the main loop of the IIPM, and the loop within the MINRES method as the inner loop of the IIPM.

In the IIPM the linear system (6) is solved with less accuracy in the initial iterations as depicted by the decreasing function η^k. Therefore, direct methods, which solve the linear system exactly, are not applicable here. Since $\mathcal{A}^k$ is banded, symmetric but indefinite, the minimum residual (MINRES) iterative method is proposed [12]. We focus on an *inexact* IPM, because the total number of floating-point operations needed to find the solution to (4) can be reduced significantly by using an inexact method to solve (6),

Algorithm 1 Inexact Interior Point Method (IIPM)

Input:
- H, F, G, f, d
- Initial guess $\theta^0, \nu^0, \phi^0 > 0, s^0 > 0, \sigma \geq 1$
- Tolerance $\epsilon > 0$

Output: Optimized θ.

Algorithm:

1: Set $k = 0$, and compute $\mu^0 := \frac{(\phi^0)^T s^0}{n_i}$ and $e_{tol}^0 := \|b^0\|$
2: **while** $\mu^k < \epsilon$ and $e_{tol}^k < \epsilon$ **do**
3: $\widetilde{H}^k := H + G^T W^k G$, where $W^k := \Phi^k (S^k)^{-1}$ is a positive diagonal matrix.
4: Solve

$$\mathcal{A}^k \mathbf{x}^k = b^k \quad (6)$$

with a preconditioned MINRES method with relative residual tolerance

$$\eta^k := \max \left\{ \min \left\{ \frac{1}{(k+1)^\sigma}, \|b^k\| \right\}, \epsilon \right\}, \quad (7)$$

where $\mathcal{A}^k, \mathbf{x}^k$ and b^k are defined in the Appendix.
5: Choose α^k as the largest value in $[0, 1]$ such that conditions (6.5) and (6.6) in [11] are satisfied.
6: $(\theta^{k+1}, \nu^{k+1}, \phi^{k+1}, s^{k+1}) := (\theta^k, \nu^k, \phi^k, s^k) + \alpha^k (\Delta\theta^k, \Delta\nu^k, \Delta\phi^k, \Delta s^k)$.
7: Compute $\mu^{k+1} := \frac{(\phi^{k+1})^T s^{k+1}}{n_i}$ and $e_{tol}^{k+1} := \|b^{k+1}\|$
8: Increment k by 1
9: **end while**

compared to using an exact method [7]. Note that the final output of an inexact IPM is within the same tolerance of the solution to (4) as the output of an exact IPM. The only difference is in the early iterations of the IPM — in later iterations the $\mathbf{x}^k$ from the inexact IPM converges to the $\mathbf{x}^k$ of the exact IPM.

IV. PRECONDITIONED ITERATIVE METHODS

There are two major factors in the computational cost of the IIPM in Algorithm 1, which are: calculation of the matrix $\widetilde{H}^k$ and computing the solution of (6). In this section we introduce some new techniques to reduce the cost of these two factors in the IIPM. The computational complexity of each method is measured in terms of floating-point operations (flops). A flop is defined as one addition, subtraction, multiplication or division of two floating-point numbers [13]. For simplicity of presentation, only higher order terms that contribute the most towards the computational cost of an algorithm are presented in this section. However, all terms, including the lower order ones, are taken into account in the numerical results presented in Section V.

The cost of each iteration of the MINRES method is $(6n^2 + 8nm + 2m^2)N$, which is mainly due to a matrix-vector multiplication. If it runs for iteration count equal to the dimension of the system matrix $\mathcal{A}^k$, then the total cost of one iteration of solving (6) in an exact IPM is given by $(12n^3 + 22n^2m + 12nm^2 + 2m^3)N^2$, which is quadratic in the horizon length N. Numerical experiments also indicate that the computational cost of the MINRES method in

solving (6) is much higher than direct methods. However, the rate of convergence of the MINRES method can be enhanced by using a suitable preconditioner [12]. A preconditioner is a matrix $\mathcal{M}$ that approximates the matrix $\mathcal{A}^k$ and its factorization is inexpensive. In the next two subsections, we investigate two preconditioners and discuss their properties.

A. Preconditioner 1

In [14], the following preconditioner is proposed for a general class of saddle point problems:

$$\mathcal{M} := \begin{bmatrix} \widetilde{H}^k + F^T K^{-1} F & 0 \\ 0 & K \end{bmatrix}, \tag{8}$$

where $K \in \mathbf{R}^{n_e \times n_e}$ is a symmetric and positive definite weight matrix.

Theorem 1: [14] Suppose $\widetilde{H}^k$ is symmetric positive semidefinite with nullity r. Then $\zeta = 1$ is an eigenvalue of $\mathcal{M}^{-1}\mathcal{A}^k$ of algebraic multiplicity n_θ and $\zeta = -1$ is an eigenvalue of multiplicity r. The remaining $n_e - r$ eigenvalues of $\mathcal{M}^{-1}\mathcal{A}^k$ are strictly between -1 and 0 and satisfy the relation $\zeta = -\frac{\alpha}{\alpha+1}$, where α are the $n_e - r$ positive generalized eigenvalues of $\alpha \widetilde{H}^k v = F^T K^{-1} F v$.

A convenient choice for the weight matrix is $K^{-1} = \gamma I$, where $\gamma > 0$ is a controlling parameter. It follows from Theorem 1 that $n_e - r$ eigenvalues of $\mathcal{M}^{-1}\mathcal{A}^k$ are not equal to ± 1 and are given by $\zeta = -\frac{\alpha\gamma}{\alpha\gamma+1}$, where α are the generalized eigenvalues of

$$\alpha \widetilde{H} v = F^T F v. \tag{9}$$

Note that if $\gamma \to \infty$, then $\zeta \to -1$. So by choosing a larger value of γ a further clustering of eigenvalues around -1 can be obtained, which reduces the number of iterations of the MINRES method significantly.

Let $\alpha_{\min}$ be the minimum generalized eigenvalue of (9) and $\bar{\zeta}$ be the corresponding eigenvalue of $\mathcal{M}^{-1}\mathcal{A}^k$. By Theorem 1, $n_e - r$ eigenvalues lie between -1 and $\bar{\zeta}$. Since we have imposed an upper bound on ζ by choosing the controlling parameter γ, we can put an upper bound on the number of iterations of the MINRES method for a given relative residual error. Following the procedure of [12, Eq. 3.13 to Eq. 3.14], we deduce that an upper bound on the iterations of the MINRES method required for a given relative residual tolerance η^k is

$$c_1^k = \frac{2\log\left(\eta^k/2\right)}{\log\left(\frac{1-\sqrt{|\bar{\zeta}|}}{1+\sqrt{|\bar{\zeta}|}}\right)}. \tag{10}$$

The above result shows that the number of iterations of the MINRES method with preconditioner $\mathcal{M}$ depends upon the tolerance η^k and $\bar{\zeta}$, but is independent of the dimension of the matrix $\mathcal{A}^k$. Notice that when $\bar{\zeta} \to 0$, then (10) implies that $c_1^k \to \infty$. However, this is not true, because there is an upper limit on the number of the MINRES method iterations in exact arithmetic. The next theorem gives us that upper bound on the number of iterations of the MINRES method with preconditioner $\mathcal{M}$.

Theorem 2: Suppose $\zeta = 1$ is an eigenvalue of $\mathcal{M}^{-1}\mathcal{A}^k$ of algebraic multiplicity n_θ and $\zeta = -1$ is an eigenvalue of multiplicity r, then the MINRES method will take at most $n_e - r + 2$ iterations to converge in exact arithmetic.

Proof: From [12, Eq. 3.7], the upper bound on the residual of (6) at the j-th iteration of the MINRES method can be written as

$$\|r_j\|/\|r_0\| \leq \min_{p_j} \max_{i=1,\ldots,n_\theta+n_e} |p_j(\lambda_i)|, \tag{11}$$

where $r_0 = b^k$ is the initial residual, $r_j = b^k - \mathcal{A}^k \mathbf{x}^j$ is the j-th residual, $p_j(\lambda_i)$ is the polynomial of degree j, and λ_i is the i-th eigenvalue of $\mathcal{M}^{-1}\mathcal{A}^k$. Let

$$p_j(\lambda_i) = 1 + a_1\lambda_i + \cdots + a_j {\lambda_i}^j, \tag{12}$$

where $a_1, \ldots, a_j$ are the unknown coefficients. Substituting (12) in (11), we get

$$\begin{aligned} \|r_j\|/\|r_0\| \quad \leq \quad & \min_{a_1,\ldots,a_j} \max\{|1 + a_1 + \cdots + a_j|, \\ & \left|1 - a_1 + \cdots + a_k(-1)^j\right|, \\ & \left|1 + a_1\lambda_1 + \cdots + a_j\lambda_1^j\right|, \ldots, \\ & \left|1 + a_1\lambda_{n_e-r} + \cdots + a_j\lambda_{n_e-r}^j\right|\}. \end{aligned} \tag{13}$$

When $j = n_e - r + 2$, the upper bound of (13) is zero, because $a_1, \ldots, a_k$ can be uniquely determined by equating each polynomial in (13) to zero. ∎

Computational cost of factorization of $\mathcal{M}$: To compute the preconditioner, we need to compute the matrix $Y = \widetilde{H} + \gamma F^T F$. The matrix product $F^T F$ can be computed outside the main loop of Algorithm 1. The matrix $Y > 0$ has a block tridiagonal form. The block Cholesky factorization of $Y = LL^T$ is determined , where L has a lower bidiagonal structure [15]. The computational cost of computing L is $(\frac{7}{3}n^3 + 3n^2m + 3nm^2 + \frac{1}{3}m^3)N$ flops.

B. Preconditioner 2

It is observed from simulations that the condition number of the matrix $\mathcal{A}^k$ increases as the number of iterations of the IIPM increases. In this section, we discuss the mechanisms of ill-conditioning, the computational cost of calculating the matrix $\widetilde{H}^k$, and how it can be reduced, and then present a low cost preconditioner.

As the iteration number k of Algorithm 1 increases, some values of the diagonal matrix W^k become very large while others become very small. As a result, the condition number of $\widetilde{H}^k$ becomes large. We can exploit this fact to reduce the computational cost of calculating $\widetilde{H}^k$. Since W^k is diagonal, $G^T W^k G$ can be written as a weighted sum of outer products of columns of G:

$$G^T W^k G = \sum_{i=1}^{n_i} w_i^k g_i g_i^T, \tag{14}$$

where g_i represents the i-th column of G, and $w_i^k := \frac{\phi_i^k}{s_i^k}$ Note that if $s_i^k \to 0$ as $k \to \infty$, then (5c) indicates that the i-th inequality constraint is active. Let us define $\mathcal{N} :=$

$\{1, 2, \ldots, n_i\}$ and $\mathcal{N}_A := \{i \in \mathcal{N} \mid \lim_{k\to\infty} s_i^k = 0\}$. The set $\mathcal{N}_A$ contains the indices of active inequality constraints at the solution. Note that if $s_i^k \nrightarrow 0$ as $k \to \infty$ then $\phi_i^k \to 0$, because $\mu^k = \frac{(\phi^k)^T s^k}{n_i} \to 0$ as $k \to \infty$. Let

$$\mathcal{N}_I := \{i \in \mathcal{N} \mid \lim_{k\to\infty} w_i^k = 0\} = \mathcal{N} \setminus \mathcal{N}_A,$$

which contains the indices of inactive inequality constraints at the solution. We define a δ-inactive set $\mathcal{N}_I^k(\delta)$ depending upon the parameter $\delta > 0$ as $\mathcal{N}_I^k(\delta) := \{i \in \mathcal{N} \mid 0 < w_i^k < \delta\}$, and a δ-active set as $\mathcal{N}_A^k(\delta) := \mathcal{N} \setminus \mathcal{N}_I^k(\delta)$. From (14), we have

$$G^T W^k G = \sum_{i\in\mathcal{N}_A^k(\delta)} w_i^k g_i g_i^T + \sum_{i\in\mathcal{N}_I^k(\delta)} w_i^k g_i g_i^T. \quad (15)$$

In practice, for larger values of k and small δ, there is a small number of active constraints, i.e. $\text{card}(\mathcal{N}_A^k(\delta)) \ll n_i$. Similarly, for larger values of k and small δ, we have $w_i^k \approx 0 \ \forall \ i \in \mathcal{N}_I^k(\delta)$. Therefore, $\widetilde{H}^k$ can be approximated as

$$\widetilde{H}^k \approx H + \sum_{i\in\mathcal{N}_A^k(\delta)} w_i^k g_i g_i^T =: \hat{H}^k. \quad (16)$$

The computational cost of calculating $\widetilde{H}^k$ is equal to $(n^2 + 2nm + m^2)n_i$. However, the computational cost of calculating the approximation of $\widetilde{H}^k$ as given in (16) is $(n^2+2nm+m^2)c_3^k$, where $c_3^k = \text{card}(\mathcal{N}_A^k(\delta))$, and $c_3^k \ll n_i$ for larger values of k. This means that we can reduce the computational cost in calculating the matrix $\widetilde{H}^k$, if we approximate it by (16), by a factor of n_i/c_3^k in each iteration k of Algorithm 1. From now on, we will replace $\widetilde{H}^k$ in all computations with its approximation $\hat{H}^k$ and the definitions of $\mathcal{A}^k$ and $\mathcal{M}$ are changed accordingly and denoted $\hat{\mathcal{A}}^k$ and $\hat{\mathcal{M}}$, respectively.

We will now exploit the observation that, for larger values of k, the last matrix in (16) is of low rank relative to its size. Hence, consider the new preconditioner

$$\mathcal{P} := \begin{bmatrix} H + \gamma F^T F & 0 \\ 0 & \frac{1}{\gamma} I \end{bmatrix}. \quad (17)$$

Since $\mathcal{P}$ is composed of matrices H and F, which are independent of any varying parameter in the main loop of Algorithm 1, its Cholesky factorization can be done outside the loop, hence does not add to the computational cost inside the loop of the IIPM. The next two theorems, which are quite general for a low rank update, are used to get a limit on the number of MINRES iterations with preconditioner $\mathcal{P}$.

Theorem 3: Let $\mathbf{B} = I + \Delta_{\mathbf{B}}$ be a symmetric matrix of size $n_{\mathbf{B}} \times n_{\mathbf{B}}$ with $\text{rank}(\Delta_{\mathbf{B}}) = r_{\Delta_{\mathbf{B}}} < n_{\mathbf{B}}$, then the MINRES method, when solving a linear system with coefficient matrix $\mathbf{B}$, terminates in at most $r_{\Delta_{\mathbf{B}}}+1$ iterations.

Proof: Since $\mathbf{B} = I + \Delta_{\mathbf{B}}$ with $\text{rank}(\Delta_{\mathbf{B}}) = r_{\Delta_{\mathbf{B}}}$, therefore $\mathbf{B}$ will have an eigenvalue at 1 of algebraic multiplicity $n_{\mathbf{B}} - r_{\Delta_{\mathbf{B}}}$. Following the procedure of Theorem 2, it can easily be shown that the MINRES method terminates in at most $r_{\Delta_{\mathbf{B}}} + 1$ iterations. ■

Theorem 4: Let $\widetilde{\mathbf{A}} = \mathbf{A} + \Delta$ where $\widetilde{\mathbf{A}}$ is a symmetric (symmetric and positive definite) matrix of size $n_{\mathbf{A}} \times n_{\mathbf{A}}$ with $\mathbf{A} > 0$ and $\text{rank}(\Delta) = r_\Delta < n_{\mathbf{A}}$. The MINRES (conjugate gradient (CG)) method with preconditioner $\mathbf{A}$, when solving a linear system with coefficient matrix $\tilde{\mathbf{A}}$, will terminate in at most $r_\Delta + 1$ iterations.

Proof: Consider the following system of linear equations:

$$(\mathbf{A} + \Delta)\bar{x} = \bar{b}. \quad (18)$$

Let $\mathbf{A} = \mathbf{L}\mathbf{L}^T$ be the Cholesky factorization of $\mathbf{A}$. Since $\mathbf{A}$ is taken as preconditioner, we effectively want to solve the following system without preconditioner

$$\left(I + \widetilde{\Delta}\right)\widetilde{x} = \widetilde{b}, \quad \bar{x} = (\mathbf{L}^{-1})^T\widetilde{x} \quad (19)$$

where $\widetilde{\Delta} := \mathbf{L}^{-1}\Delta(\mathbf{L}^{-1})^T, \widetilde{b} := \mathbf{L}^{-1}\bar{b}$, then

$$\text{rank}(\widetilde{\Delta}) \leq \min\left(\text{rank}(\mathbf{L}^{-1}), \text{rank}(\Delta)\right) = r_\Delta.$$

Following Theorem 3 for the MINRES method ([15, Thm 10.2.5] for the CG method), we get that the linear system (19) can be solved with the MINRES (CG) method in at most $r_\Delta+1$ iterations. This means that the MINRES (CG) method with preconditioner $\mathbf{A}$ terminates in at most $r_\Delta + 1$ iterations for the perturbed system (18). ■

The problem of finding the solution of the perturbed system (18), using the fact that the factorization of $\mathbf{A}$ is available in advance, is known as the low rank update problem. The computational cost of solving linear system (18) can be significantly reduced by using low rank update methods if the rank of the perturbed matrix Δ is sufficiently small. In the literature, there are two methods that are usually used for low rank update problems [15]. These are the Sherman-Morrison-Woodbury (SMW) formula and update of Cholesky factorization. For dense matrices, the computational complexity of our proposed method (Theorem 4), SMW formula, and Cholesky update is $\mathcal{O}(r_\Delta n_{\mathbf{A}}^2)$. For sparse matrices, the density of the non-zero elements in the coefficient matrix increases after the update of Cholesky factorization, which requires more memory. However, in our proposed method, we only need to store the resultant vector of the matrix-vector product, hence requires less memory and is therefore a better option for sparse matrices. For banded matrices, as in Algorithm 1, the computational complexity of our proposed method, the SMW formula, and Cholesky update is $\mathcal{O}(b_w r_\Delta n_{\mathbf{A}})$, where b_w is the bandwidth of $\widetilde{\mathbf{A}}$. However, as discussed in Section I, our iterative method is more attractive from a hardware point of view.

The following corollary can easily be deduced from Theorem 4.

Corollary 1: Suppose that $\hat{H}^k = H + \Delta_H$, where

$$\Delta_H := \sum_{i\in\mathcal{N}_A^k(\delta)} w_i g_i g_i^T \text{ and } \text{rank}(\Delta_H) = \text{card}(\mathcal{N}_A^k(\delta)) = c_3^k$$

then a preconditioned MINRES method with preconditioner $\mathcal{P}$ terminates in at most c_3^k+1 iterations, when solving a linear system with coefficient matrix $\hat{\mathcal{M}}$.

Theorem 5: A preconditioned MINRES method with preconditioner $\mathcal{P}$ terminates in at most $c_1^k + c_3^k + 1$ iterations, when solving a linear system with coefficient matrix $\hat{\mathcal{A}}^k$.

Proof: We effectively want to solve the following linear system

$$\mathcal{P}^{-1}\hat{\mathcal{A}}^k\mathbf{x}^k = \mathcal{P}^{-1}b^k \text{ or } \mathcal{P}^{-1}\hat{\mathcal{M}}\hat{\mathcal{M}}^{-1}\hat{\mathcal{A}}^k\mathbf{x}^k = \mathcal{P}^{-1}b^k$$

or following two linear systems:

$$\mathcal{P}^{-1}\hat{\mathcal{M}}y = \mathcal{P}^{-1}b^k, \quad (20a)$$
$$\hat{\mathcal{M}}^{-1}\hat{\mathcal{A}}^k\mathbf{x}^k = \hat{\mathcal{M}}^{-1}\hat{\mathcal{M}}y. \quad (20b)$$

Following Corollary 1, the linear system (20a) can be solved for y in at most $c_3^k + 1$ MINRES iterations. Similarly following (10), the linear system (20b) can be solved for $\mathbf{x}^k$ in at most c_1^k MINRES iterations. ■

C. Implementation Scheme

We propose two preconditioned MINRES methods, which we call P-MINRES-1 and P-MINRES-2.

The computational cost of an IPM per iteration with P-MINRES-1, which uses only preconditioner $\hat{\mathcal{M}}$ is given in the Table I. In the third factor, the constant c_1^k is the number of iterations required by the MINRES algorithm for a given tolerance $\eta^k < \epsilon$, as given in (10). The computational cost of this method depends significantly upon c_1^k, but we can make it small enough by choosing a suitable controlling parameter γ.

In P-MINRES-2 both preconditioners $\hat{\mathcal{M}}$ and $\mathcal{P}$ are used. In initial iterations of the inner loop of Algorithm 1, the preconditioner $\mathcal{P}$ is used. It is observed from numerical experiments that in later iterations of the main loop, sometimes the MINRES method takes a large number of iterations before it convergences to the desired accuracy and even higher than its limit as defined in Theorem 5. This is due to the finite precision effects of floating point arithmatic, which becomes important in the case when the condition number of $\hat{\mathcal{A}}^k$ is high. In that case, we switch to the preconditioner $\hat{\mathcal{M}}$ when the following condition holds:

$$c_4^{k-1} > \frac{\frac{7}{3}n^3 + 3n^2m + 3nm^2 + \frac{1}{3}m^3}{12n^2 + 18nm + 6m^2} \text{ or } \|b^k\| > \beta\|b^{k-1}\|$$

where $0 < \beta < 1$. The cost of an IPM per iteration in this method is given in Table I, where $c_2^k = 1$ if preconditioner $\hat{\mathcal{M}}$ is used, and 0 if $\mathcal{P}$ is used, and c_4^k denotes the number of MINRES iterations, which is bounded by $c_1^k + c_3^k + 1$ (Theorem 5).

V. NUMERICAL RESULTS

Consider a system of $q = \frac{n}{2} \geq 4$ equal masses connected by springs and to walls at ends. The mass of each block is 1 kg and the spring constant of each spring is taken as 1 N/m. There is no damping. There are 4 actuators connected to the first 4 masses and each can exert a maximum force of ± 0.5. The displacements of the masses are restricted to ± 4. This continuous-time state-space system is transformed into a discrete-time system using a sample time of 0.5 sec. The objective is to regulate the displacements with the given constraints on displacements and control inputs. The regulator tuning matrices are taken as $R = I, M = 0$, and $Q = [I_p\ 0]^T[I_p\ 0]$, where $p = n/2$ and the states have been ordered such that first p states describe the displacements. The matrix P satisfies the discrete-time algebraic Riccati equation. A number of simulations is carried out with initial conditions $x = 3.5[1\ 1\ 0\ \cdots\ 0]^T, \theta^0 = \mathbf{1}_{n_\theta}, \nu^0 = \mathbf{1}_{n_e}, \phi^0 = s^0 = \mathbf{1}_{n_i}, \epsilon = 10^{-3}, \delta = 0.5, \gamma = 10^7, \sigma = 4, \beta = 0.87$.

TABLE I
COST OF AN IPM PER ITERATION

Method	Flops
Riccati Recursion [5]	$(3n^3 + 6n^2m + 3nm^2 + \frac{1}{3}m^3)N +$ $l(n^2 + 2nm + m^2)(N + 1)$
P-MINRES-1	$(\frac{7}{3}n^3 + 3n^2m + 3nm^2 + \frac{1}{3}m^3)N +$ $l(n^2 + 2nm + m^2)(N + 1) +$ $(12n^2 + 18nm + 6m^2)Nc_1^k$
P-MINRES-2	$(\frac{7}{3}n^3 + 3n^2m + 3nm^2 + \frac{1}{3}m^3)Nc_2^k + (n^2 +$ $2nm + m^2)c_3^k + (12n^2 + 18nm + 6m^2)Nc_4^k$

Fig. 1(a) indicates that the rate of convergence of P-MINRES-1 and P-MINRES-2 is much higher than the unpreconditioned MINRES method. It is also observed in some cases that the unpreconditioned MINRES fails to converge and the solution never reaches the desired accuracy, due to the high condition number of matrix $\hat{\mathcal{A}}^k$. This shows that an iterative method without a preconditioner is not a good option for IIPMs. Fig. 1(b) indicates that the number of δ-active inequality constraints decreases as the iteration k of Algorithm 1 increases.

To see the growth of computational cost with the number of states n, simulations are carried out with fixed inputs m and horizon length N. Fig. 1(c) shows that the cost of P-MINRES-2 is less than the Riccati recursion method by roughly 25% to 50%. The plots of $n^2/250$ and $n^3/3500$ are also plotted for comparison. Note that P-MINRES-2 roughly scales with $\mathcal{O}(n^2)$. Secondly, keeping n and m fixed, simulations are carried out for varying N and results are plotted in Fig. 1(d). Let $c_i := \sum_{k=1}^{k=k_{\max}} c_i^k/k_{\max}$ denote the average of c_i, $\forall\ i = 1, 2, 3, 4$, where $k_{\max}$ is the maximum number of iterations taken by Algorithm 1 to terminate with the desired accuracy ϵ. Fig. 2(a) and Fig. 2(b) indicate that the average number of MINRES iterations in P-MINRES-1 (c_1) is roughly fixed, and the growth of the average number of MINRES iterations in P-MINRES-2 (c_4) with n is also not significant. Fig. 2(c) and Fig. 2(d) show that c_3 scales linearly with n and N.

VI. CONCLUSIONS

To reduce the computational cost in predictive control, so that it can be applied to fast systems, an efficient method to solve the resulting finite horizon optimal control problem (FHOCP) is presented. An inexact interior point method with an iterative linear solver (MINRES) is proposed. The use of inexact and iterative methods is motivated by the fact that these methods are attractive from a hardware point

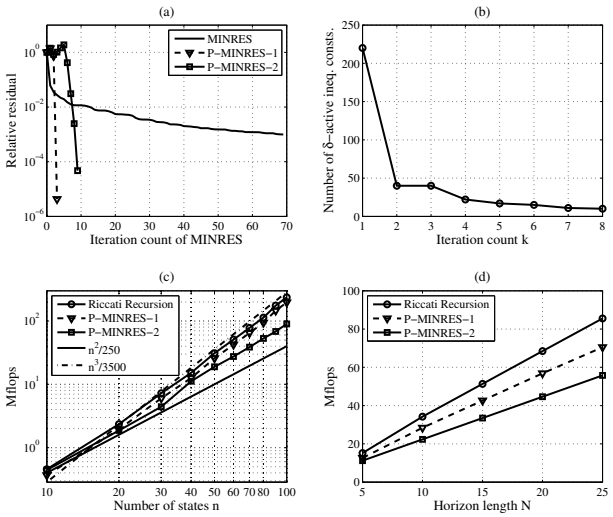

Fig. 1. (a) Comparison of MINRES, P-MINRES-1 and P-MINRES-2. (b) Decay of number of δ-active inequality constraints with main loop iteration k. The simulation results for (a) and (b) are computed with $n = 30, m = 4$ and $N = 5$. (c) Growth of number of flops with number of states n, where $m = 4$ and $N = 5$. (d) Growth of number of flops with N, where $n = 40$ and $m = 4$.

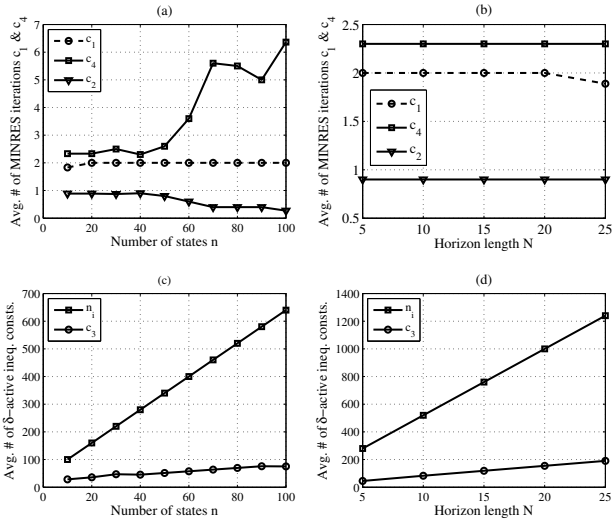

Fig. 2. (a) Growth of c_1, c_4 and c_2 with number of states n, where horizon length $N = 5$, and inputs $m = 4$. (b) Growth of c_1, c_4 and c_2 with N, where $n = 40$ and $m = 4$. (c) Growth of average number of δ-active inequality constraints c_3 with n, where $N = 5$ and $m = 4$. (d) Growth of c_3 with N, where $n = 40$ and $m = 4$. For comparison the total number of inequality constraints $n_i = l(N + 1)$ is also shown in (c) and (d).

of view [9]. The convergence of the MINRES method is enhanced with the development of efficient preconditioners.

In Section IV, we have proved a result based on the MINRES (conjugate gradient) method for the solution of a perturbed linear system with a symmetric (symmetric and positive definite) coefficient matrix. This result is quite general in the sense that it can be used for any low rank update of a linear system. This is a good alternative to its competitors the Sherman-Morrison-Woodbury formula and the update of Cholesky factorization, particularly for sparse matrices.

The computational cost involved in factorization of the preconditioner $\hat{\mathcal{M}}$, which is used in P-MINRES-1 and in latter iterations of P-MINRES-2 is quite high. Further investigations can be made to improve it.

APPENDIX

$$H := \begin{bmatrix} I_N \otimes \widetilde{Q} & 0 \\ 0 & P \end{bmatrix} \text{ where } \widetilde{Q} := \begin{bmatrix} Q & M \\ M^T & R \end{bmatrix}$$

$$F := \begin{bmatrix} -I_n & 0 \\ 0 & I_N \otimes [0 \quad -I_n] \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ I_N \otimes [A \quad B] & 0 \end{bmatrix}$$

$$\text{If } J_i = J, E_i = E \ \forall\, i \ \Rightarrow\ G := \begin{bmatrix} I_N \otimes [J \quad E] & 0 \\ 0 & J_N \end{bmatrix}$$

$$\text{If } d_i = \bar{d} \ \forall\, i \ \Rightarrow\ d := I_N \otimes \bar{d}, \quad f := \begin{bmatrix} -x^T & 0 \end{bmatrix}^T$$

$$\mathcal{A}^k := \begin{bmatrix} \widetilde{H}^k & F^T \\ F & 0 \end{bmatrix}, \mathbf{x}^k := \begin{bmatrix} \Delta\theta^k \\ \Delta\nu^k \end{bmatrix}, b^k := \begin{bmatrix} r_B^k \\ r_F^k \end{bmatrix}$$

$$\begin{aligned} r_B^k &:= r_H^k - G^T (S^k)^{-1} \left(\Phi^k r_G^k + r_S^k \right) \\ \Delta\phi^k &:= (S^k)^{-1} \left(\Phi^k r_G^k + r_S^k + \Phi^k G \Delta x^k \right) \\ \Delta s^k &:= (\Phi^k)^{-1} \left(r_S^k - S^k \Delta\phi^k \right) \end{aligned}$$

$$\begin{aligned} r_H^k &:= -\left(H\theta^k + F^T \nu^k + G^T \phi^k \right), r_S^k := -\Phi^k S^k \mathbf{1}_{n_i} \\ r_F^k &:= -\left(F\theta^k - f \right), r_G^k := G\theta^k - d + s^k \end{aligned}$$

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